

Blow-up in manifolds with generalized corners

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Abstract

We construct a functor from the category of manifolds with generalized corners to the category of complexes of toric monoids, and for every ‘refinement’ of the complex associated to a manifold, we show there is a unique ‘blow-up’, i.e., a new manifold mapping to the original one, which satisfies a universal property and whose complex realizes the refinement. This was inspired in part by the work of Gillam and Molcho, though we work with manifolds with generalized corners, as developed by Joyce, which have embedded boundary faces, for which the appropriate objects (i.e., complexes of monoids) are simpler than they would be otherwise (i.e., monoidal spaces in the sense of Kato).

1 Introduction

The aim of this paper is to provide a self-contained, differential geometric treatment of boundary blow-up in the category of manifolds with generalized corners. We use the term ‘blow-up’ in a general sense, as described below.

A *manifold with generalized corners*, as developed by Joyce [4], is a space M which is locally modeled on the spaces $X_P = \text{Hom}(P; \mathbb{R}_+)$ where the P are *toric monoids*. Such a space has an interior, M° , which is a smooth manifold without boundary, and boundary faces which are themselves manifolds with generalized corners. We construct a functor

$$M \mapsto \mathcal{P}_M, \quad (f : M \rightarrow N) \mapsto (f_{\natural} : \mathcal{P}_M \rightarrow \mathcal{P}_N)$$

from the category of manifolds with generalized corners and interior b-maps to the category of *monoidal complexes* [7]. These are roughly analogous to simplicial complexes, with toric monoids instead of simplices, and \mathcal{P}_M associates a toric monoid to each boundary face of M . A *refinement* of the monoidal complex \mathcal{P}_M is a morphism $\mathcal{R} \rightarrow \mathcal{P}_M$ which amounts to giving a consistent subdivision of the monoids in \mathcal{P}_M by toric submonoids.

Theorem (Thm. 3.7, 3.10).

- (i) *For a manifold with generalized corners M and a refinement $\psi : \mathcal{R} \rightarrow \mathcal{P}_M$, there exists a unique (up to diffeomorphism) **blow-up**, i.e., a manifold with generalized corners $[M; \mathcal{R}]$ and a **blow-down** map, $\beta : [M; \mathcal{R}] \rightarrow M$, such that $\beta : [M; \mathcal{R}]^\circ \rightarrow M^\circ$ is a diffeomorphism of interiors and*

$$\mathcal{P}_{[M; \mathcal{R}]} \cong \mathcal{R}, \quad \beta_{\natural} \cong \psi : \mathcal{R} \rightarrow \mathcal{P}_M.$$

- (ii) *The blow-up satisfies the following universal property: If the morphism $f_{\natural} : \mathcal{P}_N \rightarrow \mathcal{P}_M$ of monoidal complexes associated to an interior b-map $f : N \rightarrow M$ factors through $\psi : \mathcal{R} \rightarrow \mathcal{P}_M$, then f factors through a unique interior b-map*

$$\tilde{f} : N \rightarrow [M; \mathcal{R}].$$

- (iii) *If $f : N \rightarrow M$ is any interior b-map, then the pull back of $[M; \mathcal{R}]$ to N is a blow-up:*

$$N \times_M [M; \mathcal{R}] \cong [N; \mathcal{R}'], \quad \mathcal{R}' = \mathcal{P}_N \times_{\mathcal{P}_M} \mathcal{R}.$$

Blow-up (in this generalized sense) in the category, MC, of manifolds with (ordinary) corners was developed in [7], a principal result of which was the resolution of certain ‘binomial subvarieties’ inside a manifold with corners which arise when taking fiber products, among other situations. Following that work, Joyce in [4] developed the category, MGC, of manifolds with generalized corners, giving an intrinsic differential topological characterization of the natural class of objects exemplified by binomial subvarieties, and showed, among other results, that this category is closed under suitably transverse fiber products.

There is also an algebro-geometric theory [3] due to Gillam and Molcho, in which manifolds with corners arise as a natural subcategory of the category, PLDS, of ‘positive log differentiable spaces’. In this formulation, the ‘b-’ objects (i.e., b-maps, b-tangent bundles, b-differentials and so on) associated to manifolds with corners as defined by Melrose [9] are the natural ones corresponding to a ‘logarithmic structure’ on such a space, in the sense of [6, 5]. In addition to MC, the category PLDS includes MGC as a subcategory in addition to more general spaces. Gillam and Molcho extend Kato’s resolution of toric singularities [5] to this category. In this formulation, each space M is associated with a ‘monoidal space’ \overline{M} , which is a sheaf of toric monoids over M . To each suitable resolution $F \rightarrow \overline{M}$, they prove that there exists an essentially unique universal smooth space $N \rightarrow M$ with $\overline{N} \rightarrow \overline{M}$ factoring through F .

While Gillam and Molcho’s theory is very general, it is also quite abstract and heavily reliant on high level concepts from algebraic geometry. For this reason, we present here a short, self-contained, elementary treatment of blow-up in the category MGC. In contrast to Joyce, we require as part of the definition of a manifold with generalized corners that its boundary faces are embedded. Under this assumption, the monoidal space \overline{M} may be replaced by the monoidal complex \mathcal{P}_M , a simpler, essentially combinatorial object carrying the same information. (See the discussion in §3.4 for more on this point.)

Section 2 summarizes the necessary background material. We review toric monoids in §2.1, and then devote some detailed discussion to the model spaces X_P in §2.2 before reviewing manifolds (with generalized corners) in §2.3. Most of the ideas in §2.2 and §2.3 are due to Joyce, though some of our terminology and notation differs from [4].

We emphasize the structure of a manifold M as a *stratified space*, with strata given by the interiors, F° , of boundary faces $F \subseteq M$. Each such stratum sits

locally inside M as the subset $\{\star\} \times \mathbb{R}^l \subseteq X_{W(F)} \times \mathbb{R}^l$ where $W(F)$ is a fixed monoid (the ‘conormal monoid’), and \star is a canonical base point in the model space $X_{W(F)}$, which forms the fiber of the stratum.

The differential structure of M is encoded by the *b-tangent bundle* ${}^bTM \rightarrow M$, a real vector bundle of rank $\dim(M)$. Each boundary face $F \subseteq M$ supports a rank $\text{codim}(F)$ *b-normal* subbundle ${}^bNF \subseteq {}^bTM \rightarrow F$, with an underlying trivial bundle ${}^bMF \rightarrow F$ of monoids; in fact ${}^bNF = {}^bMF \otimes_{\mathbb{N}} \mathbb{R}$ and ${}^bMF \cong F \times W(F)^\vee$, where $W(F)^\vee$ is the ‘normal monoid’ dual to the conormal monoid above. If G and F are boundary faces with $G \subseteq F$, then $W(F)^\vee$ identifies naturally with a face of the monoid $W(G)^\vee$. An interior b-map $f : N \rightarrow M$ gives rise to a map ${}^bf_* : {}^bTN \rightarrow {}^bTM$ of vector bundles which respects these structures; in particular f induces compatible homomorphisms from the normal monoids associated with the faces of N to those of M .

The collections $\mathcal{P}_M = \{W(F)^\vee : F \subseteq M\}$ of normal monoids and these induced homomorphisms are the motivating examples of *monoidal complexes* and their morphisms, which are reviewed in §2.4, along with the notion of refinement. Finally, in Section 3, we develop the theory of blow-up, first for the model spaces in §3.1 and then for manifolds in §3.2, where we prove parts (i) and (ii) of the above theorem. We use Joyce’s result on fiber products in §3.3 to prove part (iii), and make some concluding remarks in §3.4.

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2 Background

2.1 Monoids

For a general discussion of monoids, see [10] or [3]. We review here the basic concepts that will be used below.

A **monoid** is a set $P = (P, +, 0)$ which is closed under an associative, commutative, unital binary operation, which we write additively unless otherwise specified. A monoid **homomorphism** $f : P \rightarrow Q$ is a map such that $f(p + p') = f(p) + f(p')$ and $f(0) = 0$. Every abelian group is a monoid, and a monoid homomorphism between groups is automatically a homomorphism of groups.

A **submonoid** $Q \subseteq P$ is a subset containing 0 which is closed under the binary operation (and thus is also a monoid). For each submonoid $Q \subseteq P$, there is a **quotient** monoid P/Q and a homomorphism

$$\pi : P \rightarrow P/Q$$

with the universal property that any homomorphism $h : P \rightarrow R$ for which $h(Q) = \{0\}$ factors through a unique homomorphism $\tilde{h} : P/Q \rightarrow R$, i.e., $h = \tilde{h} \circ \pi$. P/Q may be realized as the monoid of equivalence classes generated by the equivalence relation $p \sim p' \iff p + q = p' + q'$ for some $q, q' \in Q$.

A **unit** in P is an element p with a (necessarily unique) inverse q such that $p + q = 0$; the units form a submonoid (which is an abelian group) which we denote by P^\times . If $P^\times = \{0\}$ then we say P is **sharp**. For any P , the monoid

$$P^\sharp = P/P^\times$$

is sharp; it is called the **sharpening** of P .

2.1.1 Localization. To each monoid P we associate an abelian group P^{gp} and a homomorphism

$$\iota : P \rightarrow P^{\text{gp}}$$

where P^{gp} satisfies the universal property that any homomorphism $h : P \rightarrow G$ to an abelian group factors through a unique homomorphism $\tilde{h} : P^{\text{gp}} \rightarrow G$, i.e., $h = \tilde{h} \circ \iota$. It follows from the universal property that P^{gp} and ι are unique up to unique isomorphism. The **dimension** of P is

$$\dim(P) = \text{rank}(P^{\text{gp}}).$$

More generally, given any submonoid $S \subseteq P$, the **localization**, $S^{-1}P$, of P at S is a monoid with a homomorphism

$$\iota : P \rightarrow S^{-1}P$$

such that $\iota(s)$ is a unit for each $s \in S$. The localization has the the universal property that any homomorphism $h : P \rightarrow Q$ of monoids in which $h(S) \subseteq Q^\times$ factors through a unique homomorphism $\tilde{h} : S^{-1}P \rightarrow Q$. In the special case $S = P$, we have

$$P^{\text{gp}} = P^{-1}P.$$

The localization $S^{-1}P$ may be realized as the set of equivalence classes of pairs $[p, s]$ with respect to the equivalence relation $(p, s) \sim (p', s') \iff p + s' + q = p' + s + q$ for some $q \in Q$, with $\iota(p) = [p, 0]$.

2.1.2 Toric monoids. We say P is **toric** if it is:

- (T1) **finitely generated**, meaning there is a surjective homomorphism $\mathbb{N}^n \rightarrow P$ for some $n \in \mathbb{N}$ (and then P^{gp} is a finitely generated abelian group),
- (T2) **integral**, meaning that if $p + r = q + r$ in P , then $p = q$; equivalently, the map $\iota : P \rightarrow P^{\text{gp}}$ is injective,
- (T3) **torsion free**, meaning that $np = p + \dots + p = 0$ implies $p = 0$; equivalently, P^{gp} is a torsion free abelian group, and
- (T4) **saturated**, meaning that if $p \in P^{\text{gp}}$ with $np \in P$ for some $n \in \mathbb{N}$, then $p \in P$.

In particular, if P is toric then P^{gp} is a *lattice* (finitely generated, torsion free abelian group).

From this point on, *monoid* will mean *toric monoid* unless otherwise specified.

Remark. Toric monoids as defined here correspond to what Joyce calls ‘weakly toric’ monoids. Joyce reserves the term ‘toric’ for a sharp toric monoid.

As an alternative to the algebraic conditions (T1)–(T4), there is a more geometric characterization of toric monoids which makes them easier to visualize.

Proposition 2.1 ([4], Prop. 3.8). *A toric monoid is equivalent to the intersection of a finitely generated lattice L with a cone $C \subseteq V$, where $V = L \otimes_{\mathbb{Z}} \mathbb{R} \supset L$ is the associated real vector space, and C is convex, rational and polyhedral, (i.e., C is the convex hull of a finite number of rays generated by lattice elements). The monoid $P = C \cap L$ is sharp if and only if C contains no non-trivial subspace.*

In the setting of Proposition 2.1, P^{gp} is nothing other than the lattice L (assuming that the cone C does not lie in any proper subspace, in which case we can pass to the corresponding sublattice). Likewise, for each $S \subseteq P$, $S^{-1}P$ may be realized as the submonoid of L generated by P and the minimal sublattice containing S .

2.1.3 Faces and ideals. An **ideal** of P is a proper subset $I \subsetneq P$ such that $i + p \in I$ for all $i \in I$, $p \in P$. An ideal I is **prime** if $p + q \in I$ implies that either $p \in I$ or $q \in I$.

A **face** of P is a submonoid $S \subseteq P$ whose complement $P \setminus S$ is a prime ideal; thus S has the property that if $p + q \notin S$ then either $p \notin S$ or $q \notin S$. In the setting of Proposition 2.1, faces of $P = C \cap L$ are precisely the toric monoids given by the intersections $D \cap L$ where D is a face, in the obvious sense, of the polyhedral cone C . Faces (resp. prime ideals) are closed under intersection (resp. union), and there is a unique minimal face P^\times (corresponding to the unique maximal ideal $P \setminus P^\times$) and a unique maximal face P (corresponding to the minimal prime ideal \emptyset). We write

$$S \leq P$$

for the inclusion of a face, and to denote the (partial) order relation on faces determined by inclusion. Note that $T \leq S$ as faces of P if and only if T is a face of S .

The inclusion $S \rightarrow P$ generates an exact sequence

$$S^{\text{gp}} \rightarrow P^{\text{gp}} \rightarrow (P/S)^{\text{gp}} \cong P^{\text{gp}}/S^{\text{gp}} \quad (2.1)$$

of free abelian groups, and the **codimension** of S is

$$\text{codim}(S) = \text{rank}(P^{\text{gp}}/S^{\text{gp}}) = \dim(P) - \dim(S).$$

Proposition 2.2. *For each face $S \leq P$, the exact sequence (2.1) splits (non-canonically), giving an isomorphism*

$$S^{-1}P \cong P/S \times S^{\text{gp}}. \quad (2.2)$$

In particular, $P \cong P^\sharp \times P^\times$.

Since every face $S \leq P$ contains the minimal face P^\times , each quotient P/S is a sharp monoid.

Though it is not standard, we will make use of the following notion in §2.4. The **interior** of a monoid P is the complement

$$P^\circ = \bigcap_{S \leq P} P \setminus S$$

of all the proper faces of P . It is an ideal, but generally not a prime ideal. The interiors of the faces of P determine a partition $P = \bigsqcup_{S \leq P} S^\circ$. A homomorphism $f : Q \rightarrow P$ is an **interior homomorphism** if $f(P^\circ) \subseteq Q^\circ$, i.e., if f does not map Q into any proper face of P .

2.1.4 Duality. The **dual** of a monoid P is the monoid

$$P^\vee = \text{Hom}(P; \mathbb{N}).$$

Since units are preserved by homomorphisms and $\mathbb{N}^\times = \{0\}$, there is a natural isomorphism $P^\vee \cong (P^\sharp)^\vee$. Likewise P^\vee is sharp. Evaluation $p \mapsto \text{ev}_p \in \text{Hom}(P^\vee; \mathbb{N})$ determines a natural homomorphism $P \rightarrow (P^\vee)^\vee$ with kernel P^\times , giving an isomorphism

$$P^\sharp \cong (P^\vee)^\vee. \quad (2.3)$$

For each face $S \leq P$, define its **annihilator** by

$$S^\perp = \bigcap_{s \in S} \text{ev}_s^{-1}(0) = \{p \in P^\vee : p(s) = 0 \ \forall s \in S\} \leq P^\vee.$$

This is easily seen to be a face of P^\vee (the subsets $\text{ev}_s^{-1}(0)$ are prime ideals), and the association $S \mapsto S^\perp$ gives a codimension- and inclusion-reversing bijection between faces of P and faces of P^\vee . With respect to (2.3), we have $S^\sharp \cong (S^\perp)^\perp$. There is also a natural isomorphism $(P/S)^\vee \cong S^\perp$, which is to say that we have dual exact sequences of monoids

$$\begin{aligned} 0 &\longrightarrow S \longrightarrow P \longrightarrow P/S \longrightarrow 0 && \text{and} \\ 0 &\longleftarrow P^\vee/S^\perp \longleftarrow P^\vee \longleftarrow S^\perp \longleftarrow 0 \end{aligned}$$

i.e., the second is dual to the first, and vice versa if P is sharp.

Lemma 2.3. *If $S \leq P$ and $p \in P \setminus S$, then there exists $q \in S^\perp$ such that $q(p) \neq 0$.*

Proof. For each $s \in S^\perp$, $s^{-1}(0)$ is a prime ideal, which is to say the complement of some face T with $S \leq T$ by necessity. Then $(S^\perp)^{-1}(0) = \bigcup_{s \in S^\perp} s^{-1}(0)$ is the complement of a face which must be S by the property $(S^\perp)^\perp = S^\sharp$. By hypothesis $p \notin S = (S^\perp)^{-1}(0)$, so there is some $q \in S^\perp$ with $q(p) \neq 0$. \square

2.1.5 Examples. A basic example is $P = \mathbb{N}^n \times \mathbb{Z}^m$. Here $P^\times \cong \{0\} \times \mathbb{Z}^k$, so P is sharp if and only if $m = 0$, and $P^\sharp \cong \mathbb{N}^n$, while $P^{\text{gp}} = \mathbb{Z}^{n+m}$. The faces of P are the sets $\{(a_1, \dots, a_n, b_1, \dots, b_m) : a_i = 0, i \in I\}$ for various $I \subseteq \{1, \dots, n\}$.

A **freely generated** monoid is isomorphic to \mathbb{N}^n for some n ; more generally we say a monoid P is **smooth** if P^\sharp is freely generated; in this case P is isomorphic to $\mathbb{N}^n \times \mathbb{Z}^m$, where $n + m = \dim(P)$ and $m = \text{rank}(P^\times)$.

By the property (1), every sharp monoid may be presented (non-canonically) as a submonoid of \mathbb{N}^n by choosing generators $p_1, \dots, p_n \in P$ and imposing finitely many generating relations of the form

$$\sum_{i=1}^n a_i^j p_i = \sum_{i=1}^n b_i^j p_i, \quad j = 1, \dots, k, \quad a_i^j, b_i^j \in \mathbb{N}. \quad (2.4)$$

Using $P \cong P^\sharp \times P^\times$, any monoid may then be presented as a submonoid of $\mathbb{N}^n \times \mathbb{Z}^m$ by choosing generators

$$\{p_1, \dots, p_n, \pm q_1, \dots, \pm q_m\}, \quad \{p_i\} \in P \setminus P^\times, \quad P^\times \cong \mathbb{Z} \langle q_1, \dots, q_m \rangle, \quad (2.5)$$

with generating relations on the p_i as above.

For several examples of non-toric monoids, see Example 3.2 in [4]. Among these we highlight one which plays an important role below: consider the multiplicative monoid

$$\mathbb{R}_+ = ([0, \infty), \cdot, 1). \quad (2.6)$$

In the first place, \mathbb{R}_+ is not finitely generated, and the identity $0 \cdot a = 0$ for all $a \in \mathbb{R}_+$ implies that $\mathbb{R}_+^{\text{gp}} = \{0\}$, so that \mathbb{R}_+ is not integral. Moreover, $\mathbb{R}_+^\times = (0, \infty)$, so that \mathbb{R}_+ is not sharp.

2.2 Model spaces

To each monoid P , we associate the space

$$X_P = \text{Hom}(P; \mathbb{R}_+),$$

with \mathbb{R}_+ as in (2.6). We distinguish a set of **algebraic functions** on X_P , namely, for each $p \in P$, let

$$x_p : X_P \rightarrow \mathbb{R}_+, \quad x_p(x) = x(p).$$

Then X_P is given the weakest topology for which these algebraic functions are continuous.

Since homomorphisms preserve units, if $p \in P^\times$, it follows that $x(p) \in (0, \infty)$ for all $x \in X_P$; equivalently, x_p is a strictly positive function. If P is sharp, then there is a distinguished point $\star \in X_P$ given by the constant homomorphism $\star(p) = 0$ for all p , and each x_p vanishes at \star .

Remark. We will see below that the x_p play the role of *coordinates* on X_P ; for this reason we use the same letter x to denote both points of X_P and (with subscripts) algebraic functions. No confusion should arise from this convention.

The algebraic functions generate a smooth structure on X_P in the following sense. We say that a function $f : O \subseteq X_P \rightarrow \mathbb{R}$ defined on an open set is a **smooth function** if there exist $p_1, \dots, p_n \in P$ and $g \in \mathcal{C}^\infty(W; \mathbb{R})$, where $W = x_{p_1} \times \dots \times x_{p_n}(O) \subseteq \mathbb{R}_+^n$, such that

$$f = g \circ (x_{p_1} \times \dots \times x_{p_n}) = g(x_{p_1}, \dots, x_{p_n}). \quad (2.7)$$

The smooth functions form a sheaf of \mathbb{R} -algebras on X_P which we denote by

$$\mathcal{C}_{X_P}^\infty(\bullet) = \mathcal{C}^\infty(\bullet; \mathbb{R}).$$

Lying ‘in between’ the algebraic and smooth functions is the following notion. We say that a function $b : O \subseteq X_P \rightarrow \mathbb{R}_+$ defined on an open set is a **b-function** if it is locally algebraic up to multiplication by a smooth, strictly positive function; that is, for all $x \in O$, there is a possibly smaller neighborhood $x \in O' \subseteq O$ on which

$$b|_{O'} = h x_p, \quad \text{for some } p \in P, \quad h \in \mathcal{C}^\infty(O'; (0, \infty)). \quad (2.8)$$

Note that h and p are not uniquely determined since there may be many $q \in P$ for which $x_q|_{O'}$ is strictly positive. The b-functions form a sheaf of (non-toric) monoids which we denote by

$$\mathcal{B}_{X_P}(\bullet) = \mathcal{B}(\bullet; \mathbb{R}_+),$$

where $\mathcal{B}(O; \mathbb{R}_+)$ denotes the set of b-functions on O . It is often convenient to allow the constant function 0 (which is neither algebraic nor a b-function); we denote the resulting sheaf of monoids by

$$\overline{\mathcal{B}}_{X_P} = \mathcal{B}_{X_P} \sqcup 0,$$

with the obvious multiplication identity $0 \cdot b = 0$.

Remark. There is an injective morphism $\mathcal{B}_{X_P} \rightarrow \mathcal{C}_{X_P}^\infty$ of sheaves. If we consider not the sheaf $\mathcal{C}^\infty(\bullet; \mathbb{R})$ but rather $\mathcal{C}^\infty(\bullet; \mathbb{R}_+)$ as the *structure sheaf* of the space (see [3]), then we still have a morphism $\mathcal{B}_{X_P} \rightarrow \mathcal{C}_{X_P}^\infty$, but this has the property that $\mathcal{B}_{X_P}^\times \rightarrow (\mathcal{C}_{X_P}^\infty)^\times$ is an isomorphism. In the language of log geometry [6, 5], \mathcal{B}_{X_P} is a *logarithmic structure* on X_P .

The **interior** of X_P is the subspace

$$X_P^\circ = \text{Hom}(P; (0, \infty))$$

of monoid homomorphisms to $(0, \infty)$. Then $X_P^\circ \subseteq X_P$ is a dense open set. In fact, $(0, \infty)$ is a group, so by the universal property of P^{gp} we have

$$X_P^\circ = X_{P^{\text{gp}}} = \text{Hom}(P^{\text{gp}}; (0, \infty)) \cong (0, \infty)^{\dim(P)}. \quad (2.9)$$

Smooth functions on X_P° as defined above coincide with the usual notion of smooth functions on the manifold $(0, \infty)^{\dim(P)}$; thus every X_P has an interior which is diffeomorphic to $\mathbb{R}^{\dim(P)}$ via $\log : (0, \infty) \cong \mathbb{R}$.

Example 2.4. Every monoid homomorphism $x : \mathbb{N} \rightarrow \mathbb{R}_+$ is of the form $x(n) = a^n$ for some $a = x(1) \in \mathbb{R}_+$, and likewise $x : \mathbb{Z} \rightarrow \mathbb{R}_+$ must be of the same form for $a \in (0, \infty)$. Since the functor $\text{Hom}(\bullet; \mathbb{R}_+)$ preserves finite products, we have

$$X_{\mathbb{N}^n \times \mathbb{Z}^m} = \mathbb{R}_+^n \times (0, \infty)^m \cong \mathbb{R}_+^n \times \mathbb{R}^m,$$

which are the model spaces for manifolds with corners.

2.2.1 b-maps. While $P \mapsto X_P$ is a contravariant functor from monoids to spaces, we want to consider more general maps $X_P \rightarrow X_Q$ than those which arise from homomorphisms $Q \rightarrow P$. We say that a map $f : O \subseteq X_P \rightarrow X_Q$ defined on an open set is a **b-map** if, for every $q \in Q$, there exists some $p \in P$ and $h \in \mathcal{C}^\infty(O; (0, \infty))$ such that

$$f^*x_q = h x_p \quad \text{or} \quad f^*x_q = 0. \quad (2.10)$$

If the second case never occurs, we say f is an **interior** b-map. It follows that a b-map f is a **smooth map**, in the sense that $f^*\mathcal{C}_{X_Q}^\infty \rightarrow \mathcal{C}_O^\infty$; however this notion of smoothness, without the additional requirement (2.10), turns out to be too weak to be very useful. Thus by a **diffeomorphism** $f : O \subseteq X_P \rightarrow U \subseteq X_Q$, we mean an invertible interior b-map whose inverse, f^{-1} , is an interior b-map.

The following are easy consequences of the definitions.

Proposition 2.5.

- (i) A b-function $f : O \subseteq X_P \rightarrow \mathbb{R}_+$ is equivalent to an interior b-map to \mathbb{R}_+ , where the latter is considered as the model space $\mathbb{R}_+ = X_{\mathbb{N}}$.
- (ii) A b-map $f : O \subseteq X_P \rightarrow X_Q$ is interior if and only if $f(O \cap X_P^\circ) \subseteq X_Q^\circ$.
- (iii) Every homomorphism $Q \rightarrow P$ induces an interior b-map $X_P \rightarrow X_Q$.
- (iv) $f : O \subseteq X_P \rightarrow X_Q$ is an interior b-map if and only if it pulls back b-functions:

$$f^*\mathcal{B}_{X_Q} \rightarrow \mathcal{B}_O.$$

- (v) More generally, $f : O \subseteq X_P \rightarrow X_Q$ is a (not necessarily interior) b-map if and only if

$$f^*\overline{\mathcal{B}}_{X_Q} \rightarrow \overline{\mathcal{B}}_O.$$

From this point on, *map* will mean *interior b-map* unless otherwise specified.

Remark. The definitions above differ from [4]. Joyce calls b-maps and b-functions simply ‘smooth’, distinguishing smooth functions with target \mathbb{R}_+ from those with target \mathbb{R} or $(0, \infty)$.

While Example 2.4 gives the basic model space for a smooth monoid, we can embed a general X_P into some $\mathbb{R}_+^n \times \mathbb{R}^m$ by choosing generators and relations. The following is a straightforward consequence of the definitions.

Proposition 2.6 ([4], Prop. 3.14). *Let P be a monoid, with generators (2.5) and generating relations (2.4). Write $x_i = x_{p_i} : X_P \rightarrow \mathbb{R}_+$ and $y_i = x_{q_i} : X_P \rightarrow (0, \infty)$. Then the map*

$$x_1 \times \cdots \times x_n \times y_1 \times \cdots \times y_m : X_P \rightarrow \mathbb{R}_+^n \times (0, \infty)^m$$

is a diffeomorphism onto its image

$$\left\{ (x_1, \dots, x_n, y_1, \dots, y_m) : \prod_i x_i^{a_i^j} = \prod_i x_i^{b_i^j}, j = 1, \dots, k \right\} \subseteq \mathbb{R}_+^n \times (0, \infty)^m.$$

We refer to $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_m)$ as **coordinates** on X_P , and emphasize the fact that they depend on a choice of generators for P .

Suppose (x, y) and (x', y') are coordinates on X_P and X_Q , respectively, and $f : O \subseteq X_P \rightarrow X_Q$ is a b-map. Then

$$f^* x'_j = h'_j(x, y) \left(\prod_i x_i^{\mu_{ij}} \prod_i y_i^{\nu_{ij}} \right) = h_j(x, y) \prod_i x_i^{\mu_{ij}},$$

for some $\mu_{ij} \in \mathbb{N}$, $\nu_{ij} \in \mathbb{Z}$ and smooth $h'_j > 0$, and since y_i and y_i^{-1} are already smooth positive functions, we absorb these into $h_j > 0$. Since the y'_j are also smooth and positive, $f^* y'_j$ is just a smooth positive function $g_j(x, y)$. It follows that f has the coordinate form

$$\begin{aligned} f : (x, y) &\mapsto (h(x, y) x^\mu, g(x, y)) = (x', y'), \\ h &\in C^\infty(O; (0, \infty)^{n'}), \quad g \in C^\infty(O; (0, \infty)^{m'}), \quad \mu \in \text{Mat}(n \times n'; \mathbb{N}), \end{aligned} \quad (2.11)$$

where we use the obvious vector notation. Note that, with this convention, exponentiation and matrix multiplication are related by $(x^\mu)^\nu = x^{\mu\nu}$.

2.2.2 Boundary faces and support. For each face $S \leq P$, the inclusion $S \hookrightarrow P$ induces a surjective map $X_P \rightarrow X_S$. Conversely, X_S is embedded in X_P by a canonical section of this map; indeed, every $x \in X_S$ has a lift $\tilde{x} \in X_P$ defined by

$$\tilde{x}(p) = \begin{cases} x(p) & p \in S \\ 0 & p \in P \setminus S. \end{cases} \quad (2.12)$$

We identify X_S with its image in X_P under $x \mapsto \tilde{x}$, and refer to it as a **boundary face**. This embedding of X_S into X_P is a non-interior b-map. There is an inclusion-preserving bijection between faces of P and boundary faces of X_P .

Every $x \in X_P$ lies in the interior of a unique boundary face; indeed, $x^{-1}(0)$ is a prime ideal in P , and the **support** of x is the face

$$\text{supp}(x) = S \leq P \text{ such that } x^{-1}(0) = P \setminus S.$$

Then $x \in X_{\text{supp}(x)}^\circ$, and as a result we obtain a stratification

$$X_P = \bigsqcup_{S \leq P} X_S^\circ, \quad \overline{X_S^\circ} = X_S = \bigsqcup_{T \leq S} X_T^\circ, \quad (2.13)$$

where $\overline{X_S^\circ}$ denotes the closure of X_S° in X_P .

Note that the monoid P is not itself a local diffeomorphism invariant; it may certainly happen that an open set $O \subseteq X_P$ is diffeomorphic to an open set $U \subseteq X_Q$ even if $P \not\cong Q$ (for example, $X_P^\circ \cong X_Q^\circ$ whenever $\dim(P) = \dim(Q)$). Likewise, $\text{supp}(x)$ is not a local invariant, though it turns out that the quotient monoid $P/\text{supp}(x)$ is invariant. Denote by \mathcal{B}_x the stalk of \mathcal{B} at x . It is a generally non-toric monoid.

Lemma 2.7 ([4], §3.4). *For $x \in X_P$, the sharpening \mathcal{B}_x^\sharp is a toric monoid, and there is a canonical isomorphism*

$$\mathcal{B}_x^\sharp \cong P/\text{supp}(x). \quad (2.14)$$

Proof. The units \mathcal{B}_x^\times are the germs $[b]$ of b-functions such that $b(x) > 0$, and $\mathcal{B}_x^\sharp = \mathcal{B}_x/\mathcal{B}_x^\times$. The homomorphism $P \rightarrow \mathcal{B}_x^\sharp$, $p \mapsto [x_p]$ is surjective, since every b has the local form $b = h x_p$ for some p and $h > 0$, and hence $[b] \equiv [x_p] \pmod{\mathcal{B}_x^\times}$. Moreover, the kernel of this homomorphism is precisely $\text{supp}(x) \leq P$, since $x_p(x) = x(p) > 0$ if and only if $p \in \text{supp}(x)$. \square

In light of this result, we define the **codimension** of $x \in X_P$ by

$$\text{codim}(x) = \dim(\mathcal{B}_x^\sharp) = \text{codim}(\text{supp}(x)).$$

By Proposition 2.5, a local diffeomorphism $f : O \subseteq X_P \rightarrow U \subseteq X_Q$ induces isomorphisms $\mathcal{B}_x \cong \mathcal{B}_{f(x)}$ and $\mathcal{B}_x^\sharp \cong \mathcal{B}_{f(x)}^\sharp$. It follows that \mathcal{B}_x^\sharp and $\text{codim}(x)$ are local diffeomorphism invariants.

2.2.3 Localization and normal models. For each face $S \leq P$, the localization $\iota : P \rightarrow S^{-1}P$ induces a map

$$\iota^* : X_{S^{-1}P} \rightarrow X_P \quad (2.15)$$

of spaces. In the case $S = P$, this is the inclusion of $X_{P_{\text{gp}}} = X_P^\circ$ into X_P .

Proposition 2.8. *The map (2.15) is a diffeomorphism onto its image, which is a dense open set in X_P containing X_S° .*

Proof. The image of $X_{S^{-1}P} = \text{Hom}(S^{-1}P; \mathbb{R}_+)$ in X_P consists of those $x \in \text{Hom}(P; \mathbb{R}_+)$ such that $x(s) \neq 0$ for all $s \in S$ (which necessarily includes all $x \in X_S^\circ$). Conversely, any such $x \in X_P$ extends to a unique $\tilde{x} \in X_{S^{-1}P}$ via $\tilde{x}(-s) = x(s)^{-1}$. Thus ι^* is a bijection onto its image, and it is straightforward to verify that ι^* and its inverse are interior b-maps.

To see that $\iota^*(X_{S^{-1}P})$ is open and dense, we first restate the characterization of $\iota^*(X_{S^{-1}P})$ above as the condition that $x_s > 0$ for all $s \in S$. Then we can write $\iota^*(X_{S^{-1}P})$ as the finite intersection of the open dense sets $x_s^{-1}((0, \infty))$, for a finite set of generators for S . \square

From this point on, we identify $X_{S^{-1}P}$ with its image in X_P . Using (2.2) and $X_{S^{\text{gp}}} = X_S^{\circ}$ we have

$$X_{S^{-1}P} \cong X_{P/S} \times X_S^{\circ},$$

and X_S° sits in $X_{S^{-1}P}$ as the subset $\{\star\} \times X_S^{\circ}$. We call $X_{P/S}$ the **normal model** for X_S° in X_P , and refer to P/S as the **conormal monoid**, for reasons which will become clear later. Applying this to $\text{supp}(x)$ and using Lemma 2.7 we obtain

Corollary 2.9. *Each $x \in X_P$ has an open neighborhood diffeomorphic to*

$$X_{W(x)} \times \mathbb{R}^l,$$

where $W(x) = \mathcal{B}_x^{\sharp} \cong P/\text{supp}(x)$ and $l = \dim(P) - \text{codim}(x)$, in which $\text{supp}(x)$ is represented by $\{\star\} \times \mathbb{R}^l$ and x is identified with the point $(\star, 0)$.

The conormal monoid $W(x)$, the normal model space $X_{W(x)}$, and the numbers $\dim(P)$ and $\text{codim}(x)$ are all local diffeomorphism invariants.

2.3 Manifolds with generalized corners

A **manifold with generalized corners** is a second countable Hausdorff space M which is locally diffeomorphic to open sets $O \subseteq X_P$ for various P . More precisely, M is equipped with a *maximal atlas* $\{O'_a\}_{a \in A}$ of open sets $O'_a \subseteq M$ and homeomorphisms (aka *charts*) $\phi_a : O'_a \xrightarrow{\cong} O_a \subseteq X_{P(a)}$ such that the *transition functions*

$$\phi_a \phi_b^{-1} : \phi_b(O'_a \cap O'_b) \subseteq X_{P(b)} \rightarrow \phi_a(O'_a \cap O'_b) \subseteq X_{P(a)}$$

are diffeomorphisms. Since $\dim(P)$ is a local diffeomorphism invariant, it follows that the $P(a)$ for each connected component of M have the same dimension, which we define to be the **dimension** of that component; for simplicity we may assume that M is connected and then

$$\dim(M) = \dim(P(a)) \text{ for all } a.$$

A function on M is a **smooth function** (resp. **b-function**) if its compositions with the charts of M are smooth (resp. b-). These form sheaves of \mathbb{R} -algebras (resp. monoids) which we denote by \mathcal{C}_M^{∞} (resp. \mathcal{B}_M). It suffices to check these conditions with respect to a single atlas (i.e., a single open cover by charts) on M . Likewise, a **b-map** (resp. interior b-map) $f : M \rightarrow N$ between manifolds with generalized corners is a continuous map all of whose compositions with the charts of M and N are b-maps (resp. interior b-maps). Equivalently, f is a b-map if and only if $f^*(\mathcal{B}_N) \subseteq \overline{\mathcal{B}}_M = \mathcal{B}_M \sqcup 0$ and is interior if and only if $f^*(\mathcal{B}_N) \subseteq \mathcal{B}_M$.

A manifold with generalized corners has a well-defined stratification by codimension:

$$M = \bigsqcup_{0 \leq l \leq \dim(M)} S^l(M), \quad S^l(M) = \{x \in M : \text{codim}(x) = l\}. \quad (2.16)$$

Indeed, it follows from Corollary 2.9 that the $S^l(M)$ are smooth open manifolds of dimension $\dim(M) - l$ and from (2.13) that $\overline{S^l(M)} = \bigsqcup_{k \geq l} S^k(M)$.

According to Joyce, a **boundary face** of M with codimension l is a connected component, F , of the set $C_l(M)$ of pairs (x, γ) , where $x \in M$ and γ is a consistent choice of connected component of $S^l(M) \cap U$ as U ranges over sufficiently small neighborhoods of x (x itself need not be in $S^l(M)$). In local charts, F is identified with various $X_S \leq X_P$ with $\text{codim}(S) = l$. Codimension 0 boundary faces are the connected components of M .

The **interior** of a boundary face F is the set, F° , of $(x, \gamma) \in F$ such that $x \in S^l(M)$. In fact, it follows from Corollary 2.9 that each $x \in S^l(M)$ has a unique γ , so F° is simply a connected component of $S^l(M)$. Locally, F° is identified with X_S° for the various X_S . A boundary face F inherits from M the structure of a manifold with generalized corners of dimension $\dim(M) - l$, and F° is an open manifold of the same dimension.

It is often the case that the boundary faces of M are *embedded*, i.e., the map $F \rightarrow M$, $(x, \gamma) \mapsto x$ is injective. (It suffices for this to hold for boundary faces of codimension 1.) In this case a boundary face of M is simply the closure of a connected component of $S^l(M)$ in M . It is convenient to require this as part of the definition of a manifold with generalized corners, which we do from now on. We will come back to this point in §3.4.

Remark. The definition of manifolds with (ordinary) corners and b-maps [9] is recovered by requiring all the model monoids $P(a)$ to be smooth.

From this point on, *manifold* will mean *connected manifold with generalized corners and embedded boundary faces* and *map* will mean *interior b-map*, unless otherwise specified.

2.3.1 Manifolds as stratified spaces. We denote the set of boundary faces of M by $\mathcal{F}(M) = \bigsqcup_{0 \leq l \leq \dim(M)} \mathcal{F}_l(M)$, where $\mathcal{F}_l(M)$ is the set of boundary faces of codimension l , and we use the notation

$$G \leq F \iff F, G \in \mathcal{F}(M), \text{ with } G \subseteq F$$

to denote the partial order relation on boundary faces. We consider the stratification of M by boundary faces (which is finer than the stratification by codimension above):

$$M = \bigsqcup_{F \leq M} F^\circ, \quad \overline{F^\circ} = F = \bigsqcup_{G \leq F} G^\circ. \quad (2.17)$$

Corollary 2.9 implies that each $x \in F^\circ \leq M$ has a neighborhood diffeomorphic to $X_{W(x)} \times \mathbb{R}^{\dim(F)}$ for some $W(x)$. By diffeomorphism invariance of $W(x)$ and the assumption that F is connected, it follows that $W(x) = W(F)$ is independent of $x \in F^\circ$, i.e., that all $x \in F^\circ$ have the same normal model $X_{W(F)}$.

Remark. In the language of stratified spaces [11], we say that M , equipped with the stratification (2.17) is a *topologically locally trivial* stratified space,

with each stratum F° having a fixed *typical fiber* $X_{W(F)}$. The *depth* of a point $x \in M$ coincides with its codimension, and (2.16) is the depth stratification of M . It satisfies the Mather conditions, and by Proposition 2.6 admits a *smooth structure* (in the sense of stratified spaces), with respect to which the sheaf of smooth functions is equivalent to the one defined above.

Proposition 2.10. *Let $f : M \rightarrow N$ be a b-map (not necessarily interior). Then for each $F \leq M$, there is a unique $G \leq N$ with the properties that $f(F^\circ) \subseteq G^\circ$ and $f|_F : F \rightarrow G$ is an interior b-map. In particular, every b-map is a morphism of stratified spaces.*

Proof. First consider the local case of a b-map $f : O \subseteq X_P \rightarrow X_Q$. The set $q \in Q$ such that $f^*(x_q) = 0$ is a prime ideal $Q \setminus T$ for some $T \leq Q$, and it follows that $f(O) \subseteq X_T$. Since $f^*(x_t) \neq 0$ for $t \in T$, it follows that $f : O \rightarrow X_T$ is interior.

Now suppose $O \subseteq X_P$ is a chart for $O' \subseteq M$ with $F \cap O \neq \emptyset$. The stratification (2.17) restricted to O' is identified with the intersections of O with the strata $X_P = \bigsqcup_{S \leq P} X_S^\circ$; in particular $F^\circ \cap O' \cong O \cap X_S^\circ$ for some S . Applying the local result to $f|(O \cap X_S^\circ)$, it follows that f maps $F^\circ \cap O'$ into $G^\circ \cap U'$ for some $G \leq N$. Since F° and G° are smooth connected manifolds, $f : F^\circ \rightarrow G^\circ$ globally. It remains to show that $E \leq F$ implies $H \leq G$, where H is the unique boundary face such that $f : E^\circ \rightarrow H^\circ$, but since this holds locally it must hold globally as well. \square

We associate to each b-map $f : M \rightarrow N$ the map of sets

$$\mathcal{F}(f) : \mathcal{F}(M) \rightarrow \mathcal{F}(N), \quad \mathcal{F}(f)(F) = G \text{ such that } f(F^\circ) \subseteq G^\circ. \quad (2.18)$$

2.3.2 Tangent bundle. The usual notion of tangent bundle, defined via derivations on germs of smooth functions, is not particularly useful in the setting of corners, generalized or not. Melrose introduced the b-tangent bundle on a manifold with corners ([8],[9]), and in [4] Joyce extended this to the setting of generalized corners.

Let $x \in M$ and consider the stalk \mathcal{C}_x^∞ of \mathcal{C}_M^∞ at x . It has two monoid structures, with respect to addition and multiplication. There are two natural monoid homomorphisms

$$\exp : (\mathcal{C}_x^\infty, +, [0]) \rightarrow \mathcal{B}_x, \quad \text{i} : \mathcal{B}_x \rightarrow (\mathcal{C}_x^\infty, \cdot, [1]),$$

given by exponentiation and inclusion, respectively. The **b-tangent space** at x is the real vector space, ${}^bT_x M$, of pairs

$$(v, v') \in \text{Der}(\mathcal{C}_x^\infty; \mathbb{R}) \times \text{Hom}(\mathcal{B}_x; \mathbb{R}) \quad \text{such that} \\ v(\text{i}[b]) = b(x) v'([b]), \quad \text{for all } [b] \in \mathcal{B}_x. \quad (2.19)$$

Here v is a derivation on \mathcal{C}_x^∞ , v' is a monoid homomorphism from \mathcal{B}_x to $(\mathbb{R}, +)$ and (2.19) is consistent with these structures and with the \mathbb{R} vector

space structures on $\text{Der}(\mathcal{C}_x^\infty; \mathbb{R})$ and $\text{Hom}(\mathcal{B}_x; \mathbb{R})$. On one hand, since v is a derivation, $v([\exp f]) = \exp f(x) v([f])$, while on the other hand from (2.19), $v([\exp f]) = v(i(\exp[f])) = \exp f(x) v'(\exp[f])$. It follows that v and v' also satisfy the condition

$$v([f]) = v'(\exp[f]) \quad \text{for all } [f] \in \mathcal{C}_x^\infty. \quad (2.20)$$

Proposition 2.11 ([4], Example 3.41). *In the model space X_P , there is an isomorphism of vector spaces from $\text{Hom}(P; \mathbb{R})$ to ${}^bT_x X_P$ for any $x \in X_P$, given by*

$$\text{Hom}(P; \mathbb{R}) \ni \alpha \mapsto v = \sum_{p \in P} \alpha(p) x_p \partial_{x_p}, \quad (2.21)$$

with $v' \in \text{Hom}(\mathcal{B}_x; \mathbb{R})$ determined from v by $v'([x_p]) = \alpha(p)$ and (2.20).

In particular, ${}^bT_x X_P$ is a finite dimensional vector space with $\dim({}^bT_x X_P) = \dim(P)$.

The formal sum (2.21) and the characterization of v' mean that if $f \in \mathcal{C}_{X_P}^\infty(O)$ and $b \in \mathcal{B}_{X_P}(O)$ are given as in (2.7) and (2.8) by $f = g(x_{p_1}, \dots, x_{p_n})$ and $b = x_p h(x_{p_1}, \dots, x_{p_m})$ for smooth functions g and $h > 0$, then

$$\begin{aligned} v([f]) &= \sum_{i=1}^n \alpha(p_i) x_{p_i}(x) \partial_{x_{p_i}} g(x_{p_1}(x), \dots, x_{p_n}(x)), \\ v'([b]) &= \alpha(p) + \sum_{i=1}^m \alpha(p_i) x_{p_i}(x) \partial_{x_{p_i}} \log h(x_{p_1}(x), \dots, x_{p_m}(x)). \end{aligned}$$

It is an instructive exercise to check that (2.21) is consistent with relations between elements $p_i \in P$.

Proof. It is straightforward to check that $(v, v') \in {}^bT_x X_P$, and linearity of (2.21) is clear. The inverse map is given by $v' \mapsto \alpha$, where $\alpha \in \text{Hom}(P; \mathbb{R})$ is determined by $\alpha(p) = v'([x_p])$.

That α is recovered from (v, v') is clear. To see that (v, v') is recovered from α , suppose that $(v, v') \in {}^bT_x X_P$. Since v is a derivation means that on a smooth function $f = g(x_{p_1}, \dots, x_{p_n})$ it must act by

$$v([f]) = \sum_{i=1}^n c(x, p_i) \partial_{x_{p_i}} g(x_{p_1}(x), \dots, x_{p_n}(x)),$$

for some $c(x, p_i) \in \mathbb{R}$. To determine these coefficients, consider x_{p_i} as a smooth function and use (2.19) to deduce

$$c(x, p_i) = v([x_{p_i}]) = x_{p_i}(x) v'([x_{p_i}]) = x_{p_i}(x) \alpha(p_i).$$

The final statement follows from the fact that $\text{Hom}(P; \mathbb{R}) = \text{Hom}(P^{\text{gp}}; \mathbb{R})$ and $\dim(P) = \text{rank}(P^{\text{gp}})$. \square

The \mathbf{b} -tangent spaces form the fibers of the **b-tangent bundle** ${}^bTM \rightarrow M$. From the local characterization in Proposition 2.11, this inherits the structure of a smooth vector bundle of rank $\dim(M)$ in the category of manifolds with generalized corners, which is canonically trivialized over charts. A \mathbf{b} -map $f : M \rightarrow N$ induces a vector bundle morphism

$${}^bf_* : {}^bTM \rightarrow {}^bTN$$

via the pull-back action on the sheaves \mathcal{C}^∞ and \mathcal{B} .

2.3.3 Normal bundles. The inclusion $\iota_F : F \rightarrow M$ of a boundary face $F \in \mathcal{F}(M)$ is a non-interior \mathbf{b} -map. Thus, for every $x \in F$,

$$\iota_F^* : \mathcal{B}_{M,x} \rightarrow \overline{\mathcal{B}}_{F,x} = \mathcal{B}_{F,x} \sqcup 0$$

partitions $\mathcal{B}_x = \mathcal{B}_{M,x}$ into two submonoids, the **normal** and **tangential** elements with respect to F :

$$\mathcal{B}_x = \mathcal{N}(F, x) \sqcup \mathcal{T}(F, x), \quad \mathcal{N}(F, x) = (\iota_F^*)^{-1}(0), \quad \mathcal{T}(F, x) = (\iota_F^*)^{-1}(\mathcal{B}_{F,x}).$$

In other words, $[b] \in \mathcal{B}_x$ is normal to F if and only if there is a neighborhood U of x in F such that $b|_U = 0$, and tangential otherwise. In particular, if $x \in F^\circ$, then $\mathcal{T}(F, x) = \mathcal{B}_x^\times$ is just the submonoid of units, as follows from Corollary 2.9.

The **b-normal space** to F at x is the real vector space

$${}^bN_xF = \text{Hom}(\mathcal{B}_x/\mathcal{T}(F, x); \mathbb{R})$$

consisting of monoid homomorphisms from \mathcal{B}_x to \mathbb{R} which are trivial on tangential elements to F . The **normal monoid** to F at x is

$${}^bM_xF = \text{Hom}(\mathcal{B}_x/\mathcal{T}(F, x); \mathbb{N}) = (\mathcal{B}_x/\mathcal{T}(F, x))^\vee$$

and evidently ${}^bN_xF = {}^bM_xF \otimes_{\mathbb{N}} \mathbb{R}$.

There is a well-defined, injective linear map

$${}^bN_xF \rightarrow {}^bT_xM, \quad v' \mapsto (0, v'). \quad (2.22)$$

Indeed, v' determines a unique derivation $v \in \text{Der}(\mathcal{C}_x^\infty; \mathbb{R})$ satisfying (2.20); however, since the image $\exp(\mathcal{C}_x^\infty) \subseteq \mathcal{B}_x^\times \subseteq \mathcal{B}_x$ is entirely tangential to F , we must have $v = 0$.

For interior points $x \in F^\circ$, the normal monoid is ${}^bM_xF = (\mathcal{B}_x/\mathcal{T}(F, x))^\vee = (\mathcal{B}_x^\#)^\vee$, the dual to the conormal monoid $W(F) = \mathcal{B}_x^\#$. Using $\mathcal{T}(F, x)$ rather than \mathcal{B}_x^\times has the effect of extending this as a bundle over the non-interior points of F as well, as the next result shows.

Proposition 2.12. *For all $x \in F$, there is a natural monoid isomorphism*

$$\mathcal{B}_x/\mathcal{T}(F, x) \cong W(F) \quad (2.23)$$

In particular, ${}^bN_xF \cong \text{Hom}(W(F); \mathbb{R})$ and ${}^bM_xF \cong W(F)^\vee$.

Proof. This is a local statement, so it suffices to assume $M = X_P$ and $F = X_S$ for some $S \leq P$. Consider a b -function of the form $b = h x_p$, $h > 0$ on a neighborhood of x . By positivity, $[h] \in \mathcal{T}(X_S, x)$ so $[b] \equiv [x_p] \pmod{\mathcal{T}(X_S, x)}$. From (2.12) it follows that x_p is tangential to X_S if and only if $p \in S$ (while x_p may vanish at points in the boundary of X_S , it cannot vanish identically on a neighborhood unless $p \in P \setminus S$). It follows that $P \rightarrow \mathcal{B}_{X_P, x} / \mathcal{T}(X_S, x)$, $p \mapsto [x_p]$ is a surjective homomorphism with kernel S , so that $\mathcal{B}_{X_P, x} / \mathcal{T}(X_S, x) \cong P/S$, which is the conormal monoid for X_S . \square

From the local characterizations of ${}^bT_x M$ and ${}^bN_x F$ of Propositions 2.11 and 2.12, respectively, it follows that the map (2.22) extends to a short exact sequence

$$0 \rightarrow {}^bN_x F \rightarrow {}^bT_x M \rightarrow {}^bT_x F \rightarrow 0$$

coinciding locally (i.e., on charts) with the sequence

$$0 \rightarrow \text{Hom}(P/S; \mathbb{R}) \rightarrow \text{Hom}(P; \mathbb{R}) \rightarrow \text{Hom}(S; \mathbb{R}) \rightarrow 0$$

for $M = X_P$, $F = X_S$.

The b -normal spaces ${}^bN_x F$ (resp. monoids ${}^bM_x F$) form the fibers of the **b -normal bundle** ${}^bN F \rightarrow F$ (resp. **b -normal monoid bundle** ${}^bM F \rightarrow F$) which inherits from Proposition 2.12 the structure of a smooth vector bundle of rank $\text{codim}(F)$ (resp. bundle of monoids of dimension $\text{codim}(F)$) on F .

Proposition 2.13. *Let $f : M \rightarrow N$ be an interior b -map. Then for all $F \in \mathcal{F}(M)$, the differential ${}^b f_* : {}^b T M \rightarrow {}^b T N$ restricts to a vector bundle morphism*

$${}^b f_* : {}^b N F \rightarrow {}^b N G, \quad G = \mathcal{F}(f)(F), \quad (2.24)$$

where $\mathcal{F}(f)$ was defined in (2.18), which further restricts to a morphism of monoid bundles

$${}^b f_* : {}^b M F \rightarrow {}^b M G, \quad (2.25)$$

with the property that each ${}^b f_* : {}^b M_x F \rightarrow {}^b M_{f(x)} G$ is an interior homomorphism of monoids.

Proof. Let $x \in F$. Then

$$f^* : \mathcal{B}_{N, f(x)} \rightarrow \mathcal{B}_{M, x} \quad (2.26)$$

and we claim that f^* maps $\mathcal{T}(G, f(x))$ into $\mathcal{T}(F, x)$ and $\mathcal{N}(G, f(x))$ into $\mathcal{N}(F, x)$. By contradiction, suppose that $[b] \in \mathcal{T}(G, f(x))$ has image in $\mathcal{N}(F, x)$. This means that $b|G$ is a nontrivial (locally defined) b -function, but that $(f^*b)|F \equiv 0$, which contradicts the defining property of $G = \mathcal{F}(f)(F)$, namely that $f : F \rightarrow G$ is an interior b -map. Likewise, suppose that $[b] \in \mathcal{N}(G, f(x))$ has image in $\mathcal{N}(F, x)$. This means that $b|G = 0$ but that $(f^*b)|F$ is nontrivial, which contradicts the fact that $f(F) \subseteq G$.

It follows that f^* descends to a monoid homomorphism

$$f^* : \mathcal{B}_{N, f(x)} / \mathcal{T}(G, f(x)) \rightarrow \mathcal{B}_{M, x} / \mathcal{T}(F, x)$$

with no nontrivial face in its kernel. The latter property is equivalent to the property that the dual homomorphism, ${}^b f_* : {}^b M_x F \rightarrow {}^b M_{f(x)} G$, is an interior homomorphism. \square

If G and F are two boundary faces of M with $G \leq F$, then for all $x \in G$ we have a natural homomorphism

$${}^b M_x F \rightarrow {}^b M_x G \quad (2.27)$$

coming from the fact that $\mathcal{T}(G, x) \subseteq \mathcal{T}(F, x) \subseteq \mathcal{B}_{M, x}$.

Proposition 2.14. *For all $x \in G \leq F$, the homomorphism (2.27) is an isomorphism onto a boundary face of ${}^b M_x G$.*

As a consequence, the bundles ${}^b M F \rightarrow F$ and ${}^b N F \rightarrow F$ are canonically trivial for all $F \in \mathcal{F}(M)$.

Proof. In any chart $O \subseteq X_P$ at x , G and F coincide with X_T and X_S for some $T \leq S \leq P$, and then (2.27) is identified via Proposition 2.12 with the inclusion

$$(P/S)^\vee \cong S^\perp \leq T^\perp \cong (P/T)^\vee$$

of faces of P^\vee .

For the second statement, first note that if $\text{codim}(H) = 1$, then ${}^b M_x H = \mathbb{N}$ is the unique toric monoid of dimension 1, and since $\text{Aut}(\mathbb{N}) = 0$, ${}^b M H \cong H \times \mathbb{N}$ has a unique trivialization. For a general F , the inclusions ${}^b M_x H \rightarrow {}^b M_x F$ for all $H \in \mathcal{F}_1(M)$ with $F \leq H$ determine a labeling of the dimension 1 faces of each ${}^b M_x F$, and the transition maps for the bundle ${}^b M F$ must be consistent with these labelings. From Proposition 2.1, it is easy to see that any automorphism of a toric monoid fixing the faces of dimension 1 is trivial, so the local trivializations of ${}^b M F$ patch together to form a global trivialization. \square

It follows from this result that ${}^b M F$ may be identified with the trivial monoid bundle

$${}^b M F \cong F \times W(F)^\vee \rightarrow F,$$

and for $G \leq F$, the map (2.27), which must be independent of $x \in G$, identifies $W(F)^\vee$ with a face of $W(G)^\vee$. Putting this together with Proposition 2.13, for a map $f : M \rightarrow N$, the morphism (2.25), which is locally constant on the fibers, reduces to

$${}^b f_* \cong f \times f_{\natural} : F \times W(F)^\vee \rightarrow H \times W(H)^\vee, \quad H = \mathcal{F}(f)(F)$$

where $f_{\natural, F} : W(F)^\vee \rightarrow W(H)^\vee$ is a fixed homomorphism. This is the principal motivation for the notion of monoidal complexes, which we discuss next.

2.4 Monoidal complexes

A **monoidal precomplex** is a category \mathcal{P} whose objects are monoids and whose arrows, which we denote by

$$Q \rightarrowtail P \iff Q \xrightarrow{\cong} S \leq P, \quad (2.28)$$

are (injective) homomorphisms which factor as an isomorphism and the inclusion of a face. A precomplex is a **monoidal complex** if for each object $P \in \mathcal{P}$ and face $S \leq P$, there exists a unique object $Q \in \mathcal{P}$ and arrow $Q \rightarrowtail P$ which is an isomorphism onto S . It is sometimes convenient to identify S and Q , writing simply $Q \leq P$, but one must be careful since there may be multiple arrows $Q \rightarrowtail P$ mapping onto distinct faces. A **subcomplex** of a monoidal complex \mathcal{P} is a subcategory $\mathcal{S} \subseteq \mathcal{P}$ which is also a monoidal complex. If all monoids in a complex \mathcal{P} are sharp, then there is a unique initial object $0 \in \mathcal{P}$.

A **morphism of monoidal complexes**, denoted

$$\phi : \mathcal{P} \rightarrow \mathcal{Q},$$

is, for each $P \in \mathcal{P}$, an interior homomorphism $\phi_P : P \rightarrow Q$ (recall that this means $\phi_P(P^\circ) \subseteq Q^\circ$) for some $Q \in \mathcal{Q}$, such that all the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\phi_P} & Q \\ \uparrow & & \uparrow \\ S & \xrightarrow{\phi_S} & T \end{array}$$

commute, where the vertical maps are arrows in \mathcal{P} and \mathcal{Q} , respectively. If \mathcal{S} is a subcomplex of \mathcal{Q} , there is a canonical morphism $\mathcal{S} \rightarrow \mathcal{Q}$ with each homomorphism given by the identity. If $\mathcal{S} \subseteq \mathcal{Q}$ is a subcomplex and $\phi : \mathcal{P} \rightarrow \mathcal{S}$ a morphism, then

$$\mathcal{P}|\mathcal{S} = \{P \in \mathcal{P} : \phi_P : P \rightarrow S, \text{ for some } S \in \mathcal{S}\} \subseteq \mathcal{P}$$

is a subcomplex of \mathcal{P} and $\phi : \mathcal{P}|\mathcal{S} \rightarrow \mathcal{S}$ is a morphism.

Example 2.15. Fix a monoid P . The basic example of a monoidal complex is the set

$$\mathcal{P}_P = \{S : S \leq P\}$$

of faces of P with arrows given by the inclusion homomorphisms. Any subset $\mathcal{S} \subseteq \mathcal{P}_P$ for which $T \leq S$, $S \in \mathcal{S}$ implies $T \in \mathcal{S}$ is a subcomplex.

A homomorphism $f : P \rightarrow Q$ determines a unique morphism $\phi_f : \mathcal{P}_P \rightarrow \mathcal{P}_Q$ of monoidal complexes by the requirement that the ϕ_S be interior, and vice versa.

It is convenient to identify the monoidal complex \mathcal{P}_P with P itself, which we shall do from now on when no confusion can arise.

Remark. A monoidal complex is closely related to Kato's notion of a 'fan' [5]. See the discussion in §3.4.

We may summarize the observations at the end of the previous section in the following Theorem.

Theorem 2.16. *For every connected, finite dimensional manifold with generalized corners M , there is a monoidal complex*

$$\mathcal{P}_M = \{W(F)^\vee : F \in \mathcal{F}(M), {}^bMF \cong F \times W(F)^\vee\}$$

indexed by boundary faces of M , and every interior b -map $f : M \rightarrow N$ induces a morphism

$$f_\natural : \mathcal{P}_M \rightarrow \mathcal{P}_N.$$

The association $M \mapsto \mathcal{P}_M$, $f \mapsto f_\natural$ is a functor from the category of manifolds with generalized corners and interior b -maps to the category of monoidal complexes.

It is worthwhile to spell this out more explicitly for the model space X_P , as we will make heavy use of this below.

2.4.1 Monoidal complexes for model spaces. Let P be a monoid, not necessarily sharp. The boundary faces of X_P are the model spaces X_S for $S \leq P$, and we have

$$W(X_S)^\vee = (P/S)^\vee \cong S^\perp \leq P^\vee$$

For $T \leq S$, the inclusion $X_T \leq X_S$ of boundary faces induces the arrow $W(X_S)^\vee \rightarrow W(X_T)^\vee$ which is associated with the inclusion

$$W(X_S)^\vee \cong S^\perp \leq T^\perp \cong W(X_T)^\vee$$

of faces of P^\vee . In particular, $W(X_P)^\vee = \{0\}$ and $W(X_{P^\times})^\vee = P^\vee$. It follows that

$$\mathcal{P}_{X_P} \cong P^\vee$$

is identified with the complex of faces $\{S^\perp : S \leq P\}$ of the dual monoid P^\vee .

For an open set $O \subseteq X_P$,

$$\mathcal{P}_O = \{S^\perp : O \cap X_S \neq \emptyset\}$$

is the subcomplex of P^\vee consisting of monoids S^\perp for which O meets the boundary face X_S . By making O smaller if necessary, it is often convenient to assume that O meets a unique minimal boundary face X_S ; in particular $O \subseteq X_{S^{-1}P}$. Then by replacing P by $S^{-1}P$ if necessary, we may assume that O meets the minimal boundary face $X_{P^{\text{gp}}}$, and then

$$\mathcal{P}_O = \mathcal{P}_{X_P} = P^\vee.$$

If $f : O \subseteq X_P \rightarrow X_Q$ is a map, then $f_\natural : \mathcal{P}_O \rightarrow \mathcal{P}_{X_Q}$ may be identified with a single homomorphism

$$f_\natural : P^\vee \rightarrow Q^\vee. \tag{2.29}$$

The image of this homomorphism is contained in the face $T^\perp \leq Q^\vee$, where $X_T = \mathcal{F}(f)(O \cap X_{P^\times})$ is the smallest boundary face containing the image of the minimal boundary face of O . In particular, the image, $f(O)$, of O is contained in the open set $X_{T^{-1}Q} \subseteq X_Q$, so by replacing Q by $T^{-1}Q$ if necessary, we can assume that $T = Q^\times$, and then (2.29) is an interior homomorphism.

Finally, we are in a position to relate this to the local coordinate form for f .

Lemma 2.17. *Let $f : O \subseteq X_P \rightarrow X_Q$ be a map with $O \cap X_{P^\times} \neq \emptyset$ and $\mathcal{F}(f)(O \cap X_{P^\times}) = X_{Q^\times}$, and choose generators $\{p_i, \pm q_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ for P and $\{p'_i, \pm q'_j : 1 \leq i \leq n', 1 \leq j \leq m'\}$ for Q as in (2.5), with associated coordinates (x, y) on X_P and (x', y') on Q , with respect to which f is given in coordinates as in (2.11) by*

$$f(x, y) = (h(x, y) x^\mu, g(x, y)) = (x', y').$$

Then $\mu \in \text{Mat}(n \times n', \mathbb{N})$ is the matrix representing the dual homomorphism to (2.29) with respect to the generators $\{p_1, \dots, p_n\}$ of P^\sharp and $\{p'_1, \dots, p'_{n'}\}$ of Q^\sharp :

$$f_\sharp^\vee : Q^\sharp \rightarrow P^\sharp, \quad f_\sharp^\vee(p'_j) = \sum_{i=1}^n \mu_{ij} p_i.$$

Proof. This is a matter of unwinding the definitions. The homomorphism (2.29) is determined by

$${}^b f_* : {}^b M_{(0,y)} X_{P^\times} \cong P^\vee \rightarrow Q^\vee \cong {}^b M_{f(0,y)} X_{Q^\times}$$

for any point $(0, y) \in X_{P^\times}$ in the minimal boundary face. Then we may identify ${}^b M_{(0,y)} X_{P^\times} \cong (P^\sharp)^\vee$ with the dual stalk $(\mathcal{B}_{(0,y)}^\sharp)^\vee$ and likewise identify ${}^b M_{f(0,y)} X_{Q^\times} \cong (Q^\sharp)^\vee$ with $(\mathcal{B}_{f(0,y)}^\sharp)^\vee$. The dual of the homomorphism ${}^b f_*$ is just the pull-back

$$f^* : \mathcal{B}_{f(y,0)}^\sharp \rightarrow \mathcal{B}_{(0,y)}^\sharp,$$

sending $[x_{p'_j}]$ to $[h'_j \prod_i x_{p_i}^{\mu_{ij}}] = \prod_i [x_{p_i}]^{\mu_{ij}}$, and we recall that the isomorphism $\mathcal{B}_{(0,y)}^\sharp \cong P^\sharp$ is given by identifying $[x_p]$ with p . \square

2.4.2 Refinement. A **saturated refinement** is a morphism $\psi : \mathcal{R} \rightarrow \mathcal{P}$ of monoidal complexes with the following properties:

(R1) $\psi_R : R \rightarrow P$ is injective for all $R \in \mathcal{R}$, and

(R2) for each $P \in \mathcal{P}$, there is a partition

$$P^\circ = \bigsqcup_{R \rightarrow P} \psi_R(R^\circ)$$

of the interior of P into the images of the interiors of all the $R \in \mathcal{R}$ mapping to P .

In other words, for any pair $R_1, R_2 \in \mathcal{R}$ mapping to $P \in \mathcal{P}$, their images in P may only intersect at a mutual face:

$$\psi_{R_1}(R_1) \cap \psi_{R_2}(R_2) = \psi_S(S) \subseteq P \text{ for some } S \rightarrow R_i, i = 1, 2.$$

A saturated refinement encodes the idea of consistently *subdividing* each of the monoids P of \mathcal{P} into toric submonoids meeting along mutual faces, with the consistency condition that the induced subdivision of $S \leq P$ coincides with the subdivision of Q in (2.28).

In particular, in the case the $\mathcal{P} = P$ is the complex associated to a single monoid, a refinement $\mathcal{R} \rightarrow P$ may be identified with a collection $\{R_i \subseteq P\}$ of submonoids with $\dim(R_i) = \dim(P)$ subject to the conditions that $P = \bigcup_i R_i$ and $R_1 \cap R_2 = S$ such that $S \leq R_i, i = 1, 2$.

The following is immediate.

Proposition 2.18. *Let $\mathcal{R} \rightarrow \mathcal{P}$ be a refinement and $\mathcal{S} \subseteq \mathcal{P}$ a subcomplex. Then $\mathcal{R}|_{\mathcal{S}} \rightarrow \mathcal{S}$ is a refinement.*

Remark. There is a weaker notion of not necessarily saturated refinement (see [7]), wherein the condition (R2) is replaced by an analogous condition on the polyhedral cones supporting P as in Proposition 2.1, but where the images $\psi_R(R)$ need not be saturated submonoids in P . For example, the map $\mathbb{N} \rightarrow \mathbb{N}, n \mapsto 2n$ is a refinement which is not saturated.

It may be possible to generalize the notion of blow-up developed below to non-saturated refinements, as was done for smooth refinements in [7]. However, the technical machinery needed to implement this is quite non-algebraic, and will not be considered here.

From this point on, *refinement* will mean *saturated refinement*.

3 Blow-up

3.1 Blow-up of model spaces

Fix a sharp monoid P and a refinement $\psi : \mathcal{R} \rightarrow P^\vee \cong \mathcal{P}_{X_P}$. Say $R \in \mathcal{R}$ is *maximal* if $\dim(R) = \dim(P^\vee)$.

For each maximal R , the dual homomorphism $\psi_R^\vee : P \rightarrow R^\vee$ determines an interior b-map

$$X_{R^\vee} \rightarrow X_P, \tag{3.1}$$

such that $X_{R^\vee}^\circ \rightarrow X_P^\circ$ is a diffeomorphism. We proceed below to glue together the spaces X_{R^\vee} together to form a new manifold mapping to X_P .

We identify each $R \in \mathcal{R}$ its image in P^\vee , thus

$$P^\vee = \bigcup_{R_i \text{ max'l}} R_i, \quad R_1 \cap R_2 = Q, \quad Q \leq R_i, i = 1, 2.$$

Lemma 3.1. *Let $R_1, R_2 \in \mathcal{R}$ be maximal, with $Q = R_1 \cap R_2$. Then there is a natural diffeomorphism*

$$X_{(Q^\perp)^{-1}R_1^\vee} \cong X_{(Q^\perp)^{-1}R_2^\vee} \quad (3.2)$$

between the dense open sets $X_{(Q^\perp)^{-1}R_i^\vee} \subseteq X_{R_i^\vee}$, $i = 1, 2$, which is consistent with the maps (3.1).

Proof. This follows from an isomorphism $(Q^\perp)^{-1}R_1^\vee \cong (Q^\perp)^{-1}R_2^\vee$, which, having identified R_i with their images in the lattice $L := (P^\vee)^{\text{gp}}$ containing P^\vee , takes the form of an equality of monoids

$$(Q^\perp)^{-1}R_1^\vee = (Q^\perp)^{-1}R_2^\vee \subseteq L^\vee \quad (3.3)$$

in the dual lattice $L^\vee = \text{Hom}(L; \mathbb{Z}) \cong P^{\text{gp}}$. In this lattice we have

$$\begin{aligned} R_i^\vee &= \{r \in L^\vee : r(R_i) \subseteq \mathbb{N}\}, \\ Q_i^\perp &= \{q \in L^\vee : q(R_i) \subseteq \mathbb{N}, q(Q) = 0\}, \\ (Q_i^\perp)^{\text{gp}} &= (Q^\perp)^{\text{gp}} = \{q \in L^\vee : q(Q) = 0\}, \\ (Q^\perp)^{-1}R_i^\vee &= \{r + q \in L^\vee : r(R_i) \subseteq \mathbb{N}, q(Q) = 0\}. \end{aligned}$$

(We have to distinguish between the faces $Q_i^\perp \leq R_i^\vee$ since $Q_1^\perp \neq Q_2^\perp$; however their group completions are the same.) To prove (3.3), it suffices to show that $(Q^\perp)^{-1}R_1^\vee \subseteq (Q^\perp)^{-1}R_2^\vee$ by symmetry, so consider an element s in the first set. This has the property that $s(Q) \subseteq \mathbb{N}$. Let $\{p_1, \dots, p_n\}$ be a finite set of generators for R_2 , and suppose that $\{p_1, \dots, p_k\} \subseteq R_2 \setminus Q$ with $\{p_{k+1}, \dots, p_n\} \subseteq Q$. Define the integers $l_i = s(p_i) \in \mathbb{Z}$ for $i = 1, \dots, k$. For each i , if $l_i < 0$, choose $q_i \in Q^\perp$ such that $q_i(p_i) \geq -l_i$ (such an element exists by Lemma 2.3), otherwise set $q_i = 0$. Then

$$\begin{aligned} s &= r + q \in (Q^\perp)^{-1}R_2^\vee, \text{ where} \\ r &= s + \sum_i q_i \in R_2^\vee, \quad \text{and} \quad q = s - r = -\sum_i q_i \in (Q^\perp)^{\text{gp}}. \end{aligned}$$

The consistency of (3.2) with (3.1) follows from the obvious commutativity of the two homomorphisms $P \rightarrow (Q^\perp)^{-1}R_1^\vee = (Q^\perp)^{-1}R_2^\vee$ through R_1^\vee and R_2^\vee . \square

The **blow-up** of X_P by \mathcal{R} is the push-out of the X_{R^\vee} along the sets $X_{(Q^\perp)^{-1}R^\vee}$ for maximal $R \in \mathcal{R}$:

$$[X_P; \mathcal{R}] = \left(\bigcup_{R \in \mathcal{R}} X_{R^\vee} \right) / \sim \quad (3.4)$$

where the equivalence relation \sim is generated by the diffeomorphisms (3.2): $X_{R_1^\vee} \ni x_1 \sim x_2 \in X_{R_2^\vee}$ if $x_i \in X_{(Q^\perp)^{-1}R_i^\vee}$ and they are identified by such a diffeomorphism. The **blow-down map**

$$\beta : [X_P; \mathcal{R}] \rightarrow X_P \quad (3.5)$$

is well-defined by (3.1) and Lemma 3.1.

Proposition 3.2. *The blow-up $[X_P; \mathcal{R}]$ is a manifold whose monoidal complex is isomorphic to \mathcal{R} , and the blow-down map (3.5) is an interior b -map with β_{\sharp} coinciding with the refinement morphism $\psi : \mathcal{R} \rightarrow P^\vee$. Moreover, β is a diffeomorphism from $[X_P; \mathcal{R}]^\circ$ to X_P° .*

Proof. The space $[X_P; \mathcal{R}]$ has an open cover by the charts X_{R^\vee} with diffeomorphic transition maps. To see that $[X_P; \mathcal{R}]$ is Hausdorff, it suffices to show that two points $x_i \in X_{R_i^\vee} \setminus X_{(Q^\perp)^{-1}R_i^\vee}$, $i = 1, 2$ are separated in the quotient (3.5). Note that $x_i \notin X_{(Q^\perp)^{-1}R_i^\vee}$ means that $x_i(q) = 0$ for $q \in Q_i^\perp \leq R_i^\vee$.

Since R_1 and R_2 only intersect along the mutual face Q , there is an element $q \in (Q^\perp)^{\text{gp}} \subseteq L^\vee$ such that $q(R_1) \subseteq \mathbb{N}$, $q(R_2) \subseteq -\mathbb{N}$, and $q(Q) = 0$. In other words, $q \in Q_1^\perp$ and $-q \in Q_2^\perp$. Then

$$x_1 \in U_1 = \{x_q < \varepsilon\} \subseteq X_{R_1^\vee}, \quad x_2 \in U_2 = \{x_{-q} < \varepsilon\} \subseteq X_{R_2^\vee},$$

since $x_q(x_1) = x_{-q}(x_2) = 0$. However, in $X_{(Q^\perp)^{-1}R_2^\vee} \cong X_{(Q^\perp)^{-1}R_1^\vee}$, the set U_2 is identified with the set $\{x_q > \varepsilon^{-1}\}$, so $U_1 \cap U_2 = \emptyset$ for $\varepsilon < 1$.

The boundary faces of X_{R^\vee} are the subspaces $X_{T^\perp} \subseteq X_{R^\vee}$ for all $T \leq R$, which is to say that $\mathcal{P}_{X_{R^\vee}} = \{T : T \leq R\} \subseteq \mathcal{R}$. However, if T is a mutual face of both R_1 and R_2 , then the interiors $X_{T^\perp}^\circ \subseteq X_{R_i^\vee}$ are identified by (3.2). In particular there is a bijection between $T \in \mathcal{R}$ and faces of $[X_P; \mathcal{R}]$, and it follows that $\mathcal{P}_{[X_P; \mathcal{R}]} \cong \mathcal{R}$.

That $[X_P; \mathcal{R}]^\circ \cong X_P^\circ$ follows from $X_{R^\vee}^\circ \cong X_P^\circ$ and the fact that $X_{R^\vee} \subseteq X_{(Q^\perp)^{-1}R^\vee}$ for each $Q \leq R$. \square

3.1.1 Blow-up and localization. Next we investigate how the blow-up behaves with respect to passing to the open subsets $X_{S^{-1}P} \subseteq X_P$ for $S \leq P$.

From $(P/S)^\vee \cong S^\perp \leq P^\vee$, we may regard $(P/S)^\vee \subseteq P^\vee$ as a monoidal subcomplex. The restriction of the refinement $\mathcal{R} \rightarrow P^\vee$ to the subcomplex $(P/S)^\vee$ is again a refinement, so defines a blow-up of the model space $X_{P/S}$.

Proposition 3.3. *Then the preimage of the open set $X_{S^{-1}P} \cong X_{P/S} \times X_S^\circ \subseteq X_P$ under (3.5) admits a diffeomorphism*

$$\beta^{-1}(X_{S^{-1}P}) \cong [X_{P/S}; \mathcal{R}|(P/S)^\vee] \times X_S^\circ. \quad (3.6)$$

Proposition 3.3 suggests a way to define the blow-up of X_P for a non-sharp monoid P . Namely, if P is not sharp, we define $[X_P; \mathcal{R}]$ for a refinement $\mathcal{R} \rightarrow P^\vee = (P^\sharp)^\vee$ by

$$[X_P; \mathcal{R}] = [X_{P^\sharp}; \mathcal{R}] \times_{X_{P^\times}}$$

with respect to an isomorphism $P \cong P^\sharp \times P^\times$, and then this is consistent with further localization.

Lemma 3.4. *For any $R \in \mathcal{R}$, let $T = R \cap S^\perp \leq R$ be the face given by the intersection of R with the face $S^\perp \leq P^\vee$. Then the preimage of $X_{S^{-1}P}$ under (3.1) is the space $X_{(T^\perp)^{-1}R^\vee}$, and we have a commutative diagram*

$$\begin{array}{ccc} X_{(T^\perp)^{-1}R^\vee} & \hookrightarrow & X_{R^\vee} \\ \downarrow & & \downarrow \\ X_{S^{-1}P} & \hookrightarrow & X_P \end{array}$$

Proof. The preimage of $X_{S^{-1}P} = \text{Hom}(S^{-1}P; \mathbb{R}_+)$ in X_{R^\vee} consists of those $x \in \text{Hom}(R^\vee; \mathbb{R}_+)$ such that $x \neq 0$ on the image of S in R^\vee with respect to the homomorphism $P \rightarrow R^\vee$. Since $T \subseteq S^\perp$, by duality $S \subseteq T^\perp$, and S does not lie in any proper boundary face of T^\perp since then there would be a larger $Q \leq R$ for which $Q \subseteq S^\perp$. Since $x^{-1}(0)$ is the complement of a face of R^\vee , and $x \neq 0$ on S , it follows that $x \neq 0$ on T^\perp and thus $x \in X_{(T^\perp)^{-1}R^\vee}$. \square

Proof of Proposition 3.3. It follows from Lemma 3.4, that $\beta^{-1}(X_{S^{-1}P})$ consists of the union as in (3.4) of the subspaces $X_{(T^\perp)^{-1}R^\vee} \subseteq X_{R^\vee}$ for maximal $R \in \mathcal{R}$, where $T = R \cap S^\perp \leq R$. On the other hand, if we denote $\mathcal{R}|(P/S)^\vee$ by \mathcal{T} (viewed as a set of submonoids of S^\perp), the blow-up $[X_{P/S}; \mathcal{T}]$ is determined by the gluing of the spaces X_{T^\vee} for the set of $T \in \mathcal{T}$ which are maximal, i.e., $\dim(T) = \dim(P/S) = \text{codim}(S)$.

It is easy to see that each maximal $T \in \mathcal{T}$ is a face $T \leq R$ for some maximal $R \in \mathcal{R}$, but the converse is false; for $R \in \mathcal{R}$ maximal, the corresponding $T \in \mathcal{T}$ given by $T = R \cap S^\perp$ need not be maximal. However, in this latter situation there necessarily exists some other maximal $R' \in \mathcal{R}$ with a maximal face $T' \in \mathcal{T}$, for which $T \leq T'$, and $T \leq Q$, where

$$Q = R \cap R' \subseteq P^\vee,$$

as in Lemma 3.1. It follows that $(Q^\perp)^{-1}R^\vee \subseteq (T^\perp)^{-1}R^\vee$ and therefore that $X_{(T^\perp)^{-1}R^\vee}$ is entirely contained in the subset $X_{(Q^\perp)^{-1}R^\vee}$ along which X_{R^\vee} and $X_{(R')^\vee}$ are glued.

Thus it suffices to restrict consideration to those maximal $R \in \mathcal{R}$ with a maximal $T \leq R$, $T \in \mathcal{T}$. For such R and T , we claim that $X_{(T^\perp)^{-1}R^\vee} \cong X_{T^\vee} \times X_S^\circ$.

Dualizing the maps $T \hookrightarrow R$, $S^\perp \hookrightarrow P^\vee$ and $R \hookrightarrow P^\vee$, we have a commutative diagram

$$\begin{array}{ccc} T^\perp & \hookrightarrow & R^\vee \\ \downarrow & & \downarrow \\ S & \hookrightarrow & P \end{array}$$

Passing to the localizations of S and T^\perp , and using the assumption that $\dim(T) =$

$\dim(S^\perp)$, so that $\dim(T^\perp) = \dim(S)$, we have

$$\begin{array}{ccc} (T^\perp)^{\text{gp}} & \hookrightarrow & (T^\perp)^{-1}R^\vee \\ \parallel & & \downarrow \\ S^{\text{gp}} & \longrightarrow & S^{-1}P \end{array}$$

in which the left vertical arrow is an equality.

The isomorphism $S^{-1}P \cong (P/S) \times S^{\text{gp}}$ comes from a choice of splitting of the exact sequence

$$S^{\text{gp}} \rightarrow P^{\text{gp}} \rightarrow P^{\text{gp}}/S^{\text{gp}}$$

of abelian groups, and by the above this determines a compatible isomorphism

$$(T^\perp)^{-1}R^\vee \cong T^\vee \times (T^\perp)^{\text{gp}} = T^\vee \times S^{\text{gp}}$$

with respect to which the map $X_{(T^\perp)^{-1}R^\vee} \rightarrow X_{S^{-1}P}$ is identified with the map $X_{T^\vee} \times X_S^\circ \rightarrow X_{P/S} \times X_S^\circ$, with the identity map in the second factor. \square

3.1.2 Local blow-up and b-maps. Suppose $f : O \subseteq X_P \rightarrow X_Q$ is a b-map such that O meets the minimal boundary face X_{P^\times} and $\mathcal{F}(f)(O \cap X_{P^\times}) = X_{Q^\times}$, so that $f_{\natural} : \mathcal{P}_O \rightarrow \mathcal{P}_{X_Q}$ is determined by a single homomorphism $f_{\natural} : P^\vee \rightarrow Q^\vee$ as in §2.4.1.

Lemma 3.5. *Suppose $f : O \subseteq X_P \rightarrow X_Q$ is a map as above, and $\mathcal{R} \rightarrow Q^\vee$ is a refinement. If $f_{\natural} : P^\vee \rightarrow Q^\vee$ factors through some $R \in \mathcal{R}$, then there is a unique lift of f to a map*

$$\tilde{f} : O \subseteq X_P \rightarrow [X_Q; \mathcal{R}]$$

such that $f = \beta \circ \tilde{f}$.

Proof. We will show that f lifts to factor uniquely through the map $X_{R^\vee} \rightarrow X_Q$. The existence of such a map is obtained using coordinates. Thus let $\{p_1, \dots, p_n\}$, $\{p'_1, \dots, p'_{n'}\}$ and $\{r_1, \dots, r_l\}$ be generators for P^\sharp , Q^\sharp and $R^\vee = R$, respectively. By assumption we have a commutative diagrams

$$\begin{array}{ccccc} Q^\sharp & \xrightarrow{f_{\natural}^\vee} & P^\sharp & & (Q^\sharp)^{\text{gp}} & \xrightarrow{(f_{\natural}^\vee)^{\text{gp}}} & (P^\sharp)^{\text{gp}} & & \mathbb{Z}^{n'} & \xrightarrow{\mu} & \mathbb{Z}^n \\ & \searrow \beta_{\natural}^\vee & \nearrow \psi & & \searrow (\beta_{\natural}^\vee)^{\text{gp}} & \nearrow \psi^{\text{gp}} & & \searrow \nu & \nearrow \gamma & & \\ & & R^\vee & & & (R^\vee)^{\text{gp}} & & & \mathbb{Z}^l & & \end{array}$$

where each diagram is a restriction of the latter ones. Here $\mu \in \text{Mat}(n \times n'; \mathbb{N})$ represents f_{\natural}^\vee , $\nu \in \text{Mat}(n \times l; \mathbb{N})$ represents β_{\natural}^\vee and $\gamma \in \text{Mat}(l \times n'; \mathbb{N})$ represents ψ with respect to the chosen generators, and we realize $(P^\sharp)^{\text{gp}}$, $(Q^\sharp)^{\text{gp}}$ and $(R^\vee)^{\text{gp}}$ as sublattices in \mathbb{Z}^n , $\mathbb{Z}^{n'}$ and \mathbb{Z}^l , respectively.

From Lemma 2.17 and the definition of $X_{R^\vee} \rightarrow X_P$, the maps of spaces are represented with respect to coordinates (x, y) on X_P , (x', y') on X_Q and (x'', y'') on $X_{R^\vee} \times X_{Q^\times}$ by

$$\begin{aligned} f : (x, y) &\mapsto (h(x, y) x^\mu, g(x, y)) = (x', y') \\ \beta : (x'', y'') &\mapsto ((x'')^\nu, y'') = (x', y'). \end{aligned}$$

Since $(\beta_\sharp^\vee)^{\text{gp}} : (Q^\sharp)^{\text{gp}} \rightarrow (R^\vee)^{\text{gp}}$ is an isomorphism, there exists a (not necessarily unique) $\tau : \mathbb{Z}^l \rightarrow \mathbb{Z}^{n'}$ such that $\tau\nu = 1$ on the subspace $(Q^\sharp)^{\text{gp}} \subseteq \mathbb{Z}^{n'}$. Note that τ may have negative entries. In particular, $\mu\tau = \gamma$ on $(R^\vee)^{\text{gp}}$; equivalently, $\tau^\vee\mu^\vee = \gamma^\vee$ on $(P^\vee)^{\text{gp}}$. Define the b-map $\tilde{f} : O \rightarrow X_R$ in coordinates by

$$\tilde{f} : (x, y) \mapsto ((h(x, y) x^\mu)^\tau, g(x, y)) = (h(x, y)^\tau x^\gamma, g(x, y))$$

That τ may have negative entries is not a problem since the components of h are strictly positive.

Composing with β gives

$$\begin{aligned} \beta \circ \tilde{f} : (x, y) &\mapsto ((h(x, y)^\tau x^\gamma)^\nu, g(x, y)) = (((h(x, y) x^\mu)^\tau)^\nu, g(x, y)) \\ &= (h(x, y) x^\mu, g(x, y)). \end{aligned}$$

This proves existence.

Uniqueness follows from the fact that $X_{R^\vee}^\circ \cong X_Q^\circ$, so that \tilde{f} is completely determined by f as a map from $O \cap X_P^\circ$ to $X_{R^\vee}^\circ$, and then the extension to the whole domain is unique by continuity. \square

Corollary 3.6. *Suppose $O_i \subseteq X_{P_i}$, $i = 1, 2$ are open sets with a diffeomorphism*

$$f : O_1 \xrightarrow{\cong} O_2,$$

and suppose $\mathcal{R}_i \rightarrow P_i^\vee$ are refinements such that $\mathcal{R}_1|_{\mathcal{P}_{O_1}} \rightarrow \mathcal{P}_{O_1} \xrightarrow{f_\sharp} \mathcal{P}_{O_2}$ factors through an isomorphism

$$\mathcal{R}_1|_{\mathcal{P}_{O_1}} \cong \mathcal{R}_2|_{\mathcal{P}_{O_2}}.$$

Then f lifts to a unique diffeomorphism

$$\tilde{f} : O'_1 \xrightarrow{\cong} O'_2, \quad O'_i = \beta^{-1}(O_i) \subseteq [X_{P_i}; \mathcal{R}_i], \quad i = 1, 2,$$

between the preimages of the O_i in the blown up spaces.

Proof. The composition of the blow-down and f is a b-map

$$f \circ \beta_1 : O'_1 \rightarrow X_{P_2}$$

whose morphism $\mathcal{P}_{O'_1} \cong \mathcal{R}_1|_{\mathcal{P}_{O_1}} \rightarrow \mathcal{P}_{X_{P_2}} = P_2^\vee$ factors through $\mathcal{R}_2 \rightarrow P_2^\vee$ by assumption. From Lemma 3.5, this lifts to a b-map $\tilde{f} : O'_1 \rightarrow [X_{P_2}; \mathcal{R}_2]$ with image in O'_2 . Likewise, the lift of $f^{-1} \circ \beta : O'_2 \rightarrow X_{P_1}$ is a map $\tilde{g} : O'_2 \rightarrow [X_{P_1}; \mathcal{R}_1]$ with image in O'_1 .

Observe that $f \circ \beta_1 \circ \tilde{g} = \beta_2$ and $f^{-1} \circ \beta_2 \circ \tilde{f} = \beta_1$, and since the respective identity maps on O'_1 and O'_2 are b-maps lifting β_1 and β_2 , it follows from the uniqueness part of Lemma 3.5 that \tilde{f} and \tilde{g} are inverses. \square

3.2 Global blow-up

Let $\{O'_a\}_{a \in A}$ be an atlas for a manifold M , which is to say a cover by charts $\phi_a : O'_a \xrightarrow{\cong} O_a \subseteq X_{P(a)}$; by shrinking the O'_a if necessary, we assume that each O'_a meets a unique minimal boundary face $F_a \in \mathcal{F}(M)$. (In other words, among those $F \in \mathcal{F}(M)$ for which $O'_a \cap F \neq \emptyset$, there is a unique one with codimension $\max\{\text{codim}(F) : O'_a \cap F \neq \emptyset\}$.) Replacing $P(a)$ by $S^{-1}P(a)$ if necessary, we may also assume that $F_a \cap O'_a$ is identified with the minimal boundary face $X_{P(a)^\times} \subseteq X_{P(a)}$; in particular $P(a)^\# \cong W(F_a)$.

Then M admits a canonical diffeomorphism

$$M \cong \left(\bigcup_{a \in A} O_a \right) / \sim \quad (3.7)$$

where \sim is the equivalence relation generated by transition diffeomorphisms

$$O_{ba} \subseteq X_{P(a)} \xrightarrow{\cong} O_{ab} \subseteq X_{P(b)}, \quad (3.8)$$

where $O_{ba} = \phi_a(O'_a \cap O'_b) \subseteq O_a$ and $O_{ab} = \phi_b(O'_a \cap O'_b) \subseteq O_b$.

Let $\mathcal{R} \rightarrow \mathcal{P}_M$ be a refinement. For each $F \in \mathcal{F}(M)$, we regard $W(F)^\vee \subseteq \mathcal{P}_M$ as a monoidal subcomplex, so the restriction $\mathcal{R}|W(F)^\vee$ may be identified with a refinement of $P(a)^\vee = (P(a)^\#)^\vee$ whenever $F_a = F$. Let

$$U_a = \beta_a^{-1}(O_a) \subseteq [X_{P(a)}; \mathcal{R}|W(F_a)^\vee].$$

By Corollary 3.6, the transition diffeomorphisms (3.8) lift canonically to diffeomorphisms

$$U_{ba} := \beta_a^{-1}(O_{ba}) \xrightarrow{\cong} U_{ab} = \beta_b^{-1}(O_{ab}), \quad (3.9)$$

and we define the **blow-up** of M by

$$[M; \mathcal{R}] = \left(\bigcup_{a \in A} U_a \right) / \sim, \quad \beta : [M; \mathcal{R}] \rightarrow M, \quad (3.10)$$

where \sim is the equivalence relation generated by (3.9). The **blow-down map** β in (3.10) is determined by the quotients of the blow-down maps $\beta_a : U_a \rightarrow O_a$.

Theorem 3.7. *For each refinement $\psi : \mathcal{R} \rightarrow \mathcal{P}_M$, $[M; \mathcal{R}]$ is a manifold which is well-defined up to unique diffeomorphism, with $[M; \mathcal{R}]^\circ \cong M^\circ$, $\mathcal{P}_{[M; \mathcal{R}]} \cong \mathcal{R}$ and $\beta_\natural \cong \psi : \mathcal{R} \rightarrow \mathcal{P}_M$. The blow-up has the following universal property: If $f : N \rightarrow M$ is an interior b -map such that $f_\natural : \mathcal{P}_N \rightarrow \mathcal{P}_M$ factors through ψ , then f lifts to a unique interior b -map $\tilde{f} : N \rightarrow [M; \mathcal{R}]$ such that $f = \beta \circ \tilde{f}$.*

Proof. If two points $x_1 \neq x_2 \in [M; \mathcal{R}]$ satisfy $\beta(x_1) = \beta(x_2)$, then they lie in the Hausdorff subspace U_a for some a , and if $\beta(x_1) \neq \beta(x_2)$ then they are separated by the preimage under β of separating open sets in M , so $[M; \mathcal{R}]$ is a Hausdorff space, with the structure of a manifold with generalized corners coming from the charts on the U_a .

In M , each F° has a covering by charts O'_a , such that (shrinking O'_a if necessary), $O'_a \cong O_a \subseteq X_{W(F)} \times X_{\mathbb{Z}^l}$, $l = \dim(F)$, with the transition diffeomorphisms preserving this product structure. The preimages of these charts in $[M; \mathcal{R}]$ have the form $U_a \subseteq [X_{W(F)}; \mathcal{R}|W(F)^\vee] \times X_{\mathbb{Z}^l}$, with transition diffeomorphisms again preserving the product structure. It follows that $\beta^{-1}(F^\circ) = \bigsqcup G_R^\circ$ consists of the union of the interiors of boundary faces $G_R \in \mathcal{F}([M; \mathcal{R}])$ corresponding to $R \in \mathcal{R}|W(F)^\vee$ with $R^\circ \subseteq (W(F)^\vee)^\circ$, with $\text{codim}(G_R) = \dim(R)$ and $G_R \leq G_{R'} \iff R' \leq R$. By Proposition 2.10, for each $G \in \mathcal{F}([M; \mathcal{R}])$ there is a unique $F \in \mathcal{F}(M)$ with $\beta(G^\circ) \subseteq F^\circ$, so taking the union over $F \in \mathcal{F}(M)$, it follows that $\mathcal{P}_{[M; \mathcal{R}]} \cong \mathcal{R}$.

The lifting of interior b-maps follows from the local version. Indeed, given $f : N \rightarrow M$, we may refine the atlas on N so that each chart $O_b \subseteq N$ has image in some $O_a \subseteq M$, and then the lifted map is given by patching together the lifted maps $f : O_b \rightarrow U_a \subseteq [M; \mathcal{R}]$, which are necessarily compatible by the uniqueness in Lemma 3.5.

The uniqueness of $[M; \mathcal{R}]$ up to diffeomorphism follows from this, since another choice of atlas leads to a space $[M; \mathcal{R}]'$ and a unique diffeomorphism $[M; \mathcal{R}]' \cong [M; \mathcal{R}]$. \square

3.3 Blow-up and pull back

We recall one of the main results from [4], generalizing a similar result for manifolds with ordinary corners in [7]. Suppose $f : Y \rightarrow M$ and $g : Z \rightarrow M$ are interior b-maps of manifolds with generalized corners. The maps are said to be **b-transverse** if

$${}^b f_* \oplus {}^b g_* : {}^b T_y Y \oplus {}^b T_z Z \rightarrow {}^b T_x M$$

is surjective for all $(y, z) \in Y \times Z$ such that $f(y) = g(z) = x$.

Theorem 3.8 ([4], Thm. 4.27). *If $f : Y \rightarrow M$ and $g : Z \rightarrow M$ are b-transverse maps, then the fiber product (pull back) $Y \times_M Z$ exists in the category of manifolds with generalized corners and interior b-maps. More explicitly,*

$$Y \times_M Z = \overline{\{(y, z) \in Y^\circ \times Z^\circ : f(y) = g(z)\}} \subseteq Y \times Z$$

admits a canonical structure of a manifold with generalized corners, with respect to which it satisfies the following universal property: If $k : N \rightarrow Y$ and $l : N \rightarrow Z$ are interior b-maps such that $f \circ k = g \circ l$, then there exists a unique interior b-map $h : N \rightarrow Y \times_M Z$ such that $k = \text{pr}_1 \circ h$ and $l = \text{pr}_2 \circ h$:

$$\begin{array}{ccccc}
N & & & & \\
& \searrow h & & \nearrow l & \\
& Y \times_M Z & \xrightarrow{\text{pr}_2} & Z & \\
& \downarrow \text{pr}_1 & & \downarrow g & \\
& Y & \xrightarrow{f} & M &
\end{array}$$

(Note: In the original image, there is also a curved arrow from N to Y labeled k .)

Proposition 3.9. *A blow-down map $\beta : [M; \mathcal{R}] \rightarrow M$ has the property that*

$${}^b\beta_* : {}^bT_x[M; \mathcal{R}] \xrightarrow{\cong} {}^bT_{\beta(x)}M$$

is an isomorphism for all $x \in [M; \mathcal{R}]$.

Proof. The property is local, so it suffices to consider the case $\beta : X_{R^\vee} \rightarrow X_P$ for a maximal R in a refinement of P^\vee . By Proposition 2.11, ${}^bT_x X_{R^\vee} \cong \text{Hom}(R^\vee; \mathbb{R}) = \text{Hom}((R^\vee)^{\text{gp}}; \mathbb{R})$ and ${}^bT_{\beta(x)} X_P \cong \text{Hom}(P; \mathbb{R}) = \text{Hom}(P^{\text{gp}}; \mathbb{R})$, and the linear map between them is generated by the homomorphism $P^{\text{gp}} \rightarrow (R^\vee)^{\text{gp}}$ determined by duality from $R \rightarrow P^\vee$. Since the latter is injective with $\dim(R) = \dim(P^\vee)$, the former is an isomorphism. \square

It follows that $\beta : [M; \mathcal{R}] \rightarrow M$ is b-transverse to *any* interior b-map $f : Y \rightarrow M$; in particular, the pull back

$$Y \times_M [M; \mathcal{R}] \rightarrow Y$$

is a well-defined manifold for every map $f : Y \rightarrow M$.

On the other hand, fiber products exist in the category of monoidal complexes, and the pull back of a refinement is a refinement [7]. In particular, given a refinement $\psi : \mathcal{R} \rightarrow \mathcal{P}_M$ and a map $f : Y \rightarrow M$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_Y \times_{\mathcal{P}_M} \mathcal{R} & \longrightarrow & \mathcal{R} \\ \downarrow & \searrow f_{\natural} & \downarrow \psi \\ \mathcal{P}_Y & \longrightarrow & \mathcal{P}_M \end{array}$$

of monoidal complexes, in which the vertical arrows are refinements.

Theorem 3.10. *Let $[M; \mathcal{R}]$ be the blow-up of a manifold M with respect to a refinement $\mathcal{R} \rightarrow \mathcal{P}_M$, and let $f : Y \rightarrow M$ be an interior b-map. Then there is a canonical diffeomorphism*

$$Y \times_M [M; \mathcal{R}] \cong [Y; \mathcal{P}_Y \times_{\mathcal{P}_M} \mathcal{R}] \quad (3.11)$$

between the pull back of $[M; \mathcal{R}]$ over Y and the blow-up of Y by the refinement $\mathcal{P}_Y \times_{\mathcal{P}_M} \mathcal{R}$. In other words, blow-ups pull back under arbitrary interior b-maps.

Proof. Denote the refinement $\mathcal{P}_Y \times_{\mathcal{P}_M} \mathcal{R}$ by $\mathcal{R}' \rightarrow \mathcal{P}_Y$. Suppose N is a manifold with maps to Y and $[M; \mathcal{R}]$ forming a commutative square with M , thus inducing a commutative square of complexes:

$$\begin{array}{ccc} N & \longrightarrow & [M; \mathcal{R}] \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M, \end{array} \quad \begin{array}{ccc} \mathcal{P}_N & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \mathcal{P}_Y & \longrightarrow & \mathcal{P}_M. \end{array}$$

Then \mathcal{P}_N factors uniquely through \mathcal{R}' by the universal property of the fiber product of complexes, and from the universal property of blow-up it follows

that N factors through a unique map to $[Y; \mathcal{R}']$. In other words, the manifold $[Y; \mathcal{R}']$ satisfies the same universal property as the fiber product $Y \times_M [M; \mathcal{R}]$, which is unique up to canonical diffeomorphism. \square

Remark. In the language of algebraic geometry, blow-down maps are *stable under base change*.

3.4 Commentary

The assumption that boundary faces of manifolds are embedded is necessary if one wants to work with monoidal complexes, as we have done. Indeed, the embeddedness assumption was used in Proposition 2.14 to obtain the triviality of the bundles bMF ; without this assumption it is straightforward to construct examples where the bMF are not trivial. Moreover, even if the bMF are trivial, so that one still obtains a complex \mathcal{P}_M , it may not be possible to realize a refinement by blow-up, i.e., the statement that $\mathcal{P}_{[M; \mathcal{R}]} \cong \mathcal{R}$, which depends on the embeddedness assumption, may fail.

To illustrate this last point, consider the *teardrop* (c.f. [4], Example 2.8)

$$M = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y^2 \leq x^2 - x^4\}.$$

This is a 2-dimensional manifold with (ordinary) corners having a single codimension 2 boundary face at the origin and a single, self-intersecting boundary hypersurface. Its monoidal complex is

$$\mathcal{P}_M : \{0\} \longrightarrow \mathbb{N} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \mathbb{N}^2 \quad (3.12)$$

where the single object \mathbb{N} is identified with both of the faces of \mathbb{N}^2 . By contrast, the complex $\mathcal{P}_{\mathbb{N}^2}$ is

$$\mathcal{P}_{\mathbb{N}^2} : \{0\} \begin{array}{cc} \nearrow & \mathbb{N} \\ \searrow & \mathbb{N}^2 \\ \nwarrow & \mathbb{N} \\ \nearrow & \end{array}$$

It is easy to see that there are no injective morphisms $\mathcal{P}_M \rightarrow \mathcal{P}_{\mathbb{N}^2}$, while there is an obvious morphism $\mathcal{P}_{\mathbb{N}^2} \rightarrow \mathcal{P}_M$ given by the identity on each \mathbb{N}^n . This latter morphism is a refinement, and the construction of $[M; \mathcal{P}_{\mathbb{N}^2}]$ given above goes through since it is completely local. However, since the only morphisms in the refinement are identities, we just recover M again, i.e., $[M; \mathcal{P}_{\mathbb{N}^2}] \cong M$, but $\mathcal{P}_M \not\cong \mathcal{P}_{\mathbb{N}^2}$.

To work with such spaces then, it is necessary to give up the complex \mathcal{P}_M in favor of a more complicated object.

A **monoidal space**, (Y, \mathcal{M}_Y) , as defined by Kato [5] (see also [3]), is a topological space Y equipped with a sheaf \mathcal{M}_Y of sharp monoids, and a morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ is a continuous map $f : X \rightarrow Y$ with a morphism $f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$ of sheaves. A manifold M with generalized corners admits

the structure of a monoidal space, whose underlying topological space is M , and whose sheaf of monoids is the sharpening $\mathcal{M}_M = \mathcal{B}_M^\sharp$ of the sheaf of b-functions. This sheaf has the property that if O meets a unique face F of maximal codimension, then $\mathcal{B}_M^\sharp(O) = W(F)$.

In fact, the monoidal complex of a manifold with embedded boundary faces is essentially equivalent to Kato's notion of the 'fan' associated to certain logarithmic schemes [5]. A **fan** is a monoidal space locally isomorphic to the 'affine' model space $(\mathrm{Spec}(P), \mathcal{M}_P)$. Here $\mathrm{Spec}(P) = \{F : F \leq P\}$ is the set of faces (equivalently, prime ideals) of a monoid P equipped with the (non-Hausdorff) Zariski topology generated by open sets $U_p = \{F : p \in F\}$ for $p \in P$, and \mathcal{M}_P is the sheaf of sharp monoids whose stalk at $F \in \mathrm{Spec}(P)$ is the monoid $\mathcal{M}_{P,F} = P/F$. (The concept of a fan is summed up succinctly by the analogy fan : sharp monoid :: scheme : ring.) In contrast to a general monoidal space, a fan consists of a small (typically finite) number of points; indeed, there is a bijection between the affine open sets of a fan and its points (c.f. Lemma 4.6, [1]). Certain sufficiently nice logarithmic schemes $(X, \mathcal{M}_X, \mathcal{O}_X)$ (analogous to our manifolds with embedded boundary faces) are associated to a canonical fan F via a morphism $(X, \mathcal{M}_X^\sharp) \rightarrow F$ which essentially collapses various strata (analogous to our interiors of boundary faces) down to points. In this analogy, the fan associated to a manifold with embedded boundary faces has a single point for each stratum of (2.17) and is equipped with a non-Hausdorff topology (encoding the inclusion relations between boundary faces) and a sheaf obtained from the dual sheaf $(\mathcal{B}_M^\sharp)^\vee$; in particular its monoids are dual to those in the complex \mathcal{P}_M .

That general manifolds (without embedded boundary faces) do not admit monoidal complexes can be compared to the fact that not all logarithmic schemes admit fans [1]. To define blow-up for manifolds in general, it should still be possible to explicitly patch together the local constructions in §3.1 for a suitable notion of refinement of the monoidal space $(M, \mathcal{B}_M^\sharp)$. Indeed, this is the approach taken by [3], though their approach is rather abstract. Alternatively, it may be possible to work with some kind of intermediate object which is simpler than $(M, \mathcal{B}_M^\sharp)$ but more complicated than \mathcal{P}_M (compare the notion of an 'Artin fan' [2, 1] in logarithmic algebraic geometry). We leave this for a future work.

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